Definite Integration using the Generalized Hypergeometric Functions

by

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Chairman, Departmental Committee on Graduate Students

Dedicated to my Mother, to my Father

and to all my Teachers

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ABSTRACT

A design for the definite integration of approximately fifty Special Functions is described. The Generalized Hypergeometric Functions are utilized as a basis for the representation of the members of the above set of Special Functions. Only a relatively small number of formulas that generally involve Generalized Hypergeometric Functions are utilized for the integration stage. A last and crucial stage is required in the integration process: the reduction of the Generalized Hypergeometric Function to Elementary and/or Special Functions.

The results of an early implementation which involves Laplace transforms are given and some actual examples with their corresponding timing are provided.

THESIS SUPERVISOR: Joel Moses

TITLE: Professor of Computer Science and Engineering

Chapter 1

INTRODUCTION

We present[1] a procedure for the definite integration of a class of Special Functions, the so called functions of mathematical physics. In providing this procedure, we include all the well known Special Functions - approximately fifty - that often arise in mathematical problems in experimental and theoretical physics, in mathematical astronomy and satelite theory, as well as in all branches of engineering - electrical, nuclear, naval, aero etc. .

The area of Special Functions, despite its wide applicability to problems of many areas of engineering and science, is very well known for its "chaotic state"[2]. For us, the wide applicability was the most attractive point and strongly motivated us throughout our research, while the "chaotic state" of the domain became to us a challenging target for exploration. A lot of information, encyclopedic in nature, can be found in numerous books and articles which include specific problems and methods, most of which are mainly results of particular applications. The tools that are mainly used are those provided by classical mathematical analysis [3], [4].

1. Small portions of this thesis have been copied from the author's paper "Symbolic Laplace Transforms for Special Functions" [1]

An effort towards structuring the domain of Special Functions has been employed by using Lie Algebras and Lie Groups [2], [5]. This approach is an effort to "bridge" the big gap between pure and applied mathematics much effort should be done in this area in the future.

In computer science, there has been considerable effort to compute values of some very important Special Functions with different numerical and approximating techniques [6]. From the point of view of symbolic mathematics, some cases of the Error, Beta, and Gamma functions have been employed in indefinite integration [7], and definite integration [8]. All of these packages are implemented in the symbolic manipulation system MACSYMA [9] at the M.I.T. Laboratory for Computer Science (formerly Project MAC). To the best of our knowledge there has been no other system designed for manipulation of the integral transforms or definite integration of the approximately fifty Special Functions, wherein the focus of our thesis lies.

One faces two main difficulties when dealing with the problem of definite integration of Special Functions in symbolic manipulation. First, the area of Special Functions as we have already mentioned has been acknowledged as a "chaotic area". Second, definite integration generally is a recursively unsolvable problem [10]. In our procedure we take advantage of the fact that most of the Special Functions can be considered as particular instances of the Generalized Hypergeometric Function and therefore can be integrated, using the Generalized Hypergeometric Function representation, with a table consisting of very few formulas. Besides, we were strongly influenced by the monumental work of "Bateman's Manuscript Project" [3], [11] to view a significant part of Definite Integration as particular instances of Integral Transforms.

What our thesis will try to show is not a general algorithm, but what is the best way, using currently available knowledge, to solve a large portion of the aforementioned problem in a fashion which is relatively general and computationally effective.

The results we obtained in an early implementation which did not incorporate all of the described methods in this thesis encouraged us tremendously and ultimately influenced our decisions that this was a good way to follow (Results of this early implementation with some actual examples accompanied with their timing is shown in the Appendix I). Moreover, we had the chance to utilize available machinery from the classical mathematical analysis and create new algorithmic techniques as well as new formulas (see chapter 3).

The notation we follow throughout this thesis is the traditional one and established by the Bateman Manuscript Project [3], [11].

1.1. OVERVIEW OF OUR APPROACH

In this section we provide a general overview of our scheme of definite integration. Our principal strategy for the definite integrals is to classify them as some kind of integral transform. We have been mainly concerned with the integration of the class of Special Functions. The main vehicle for the class of Special Functions is the Generalized Hypergeometric Function [12].

Definition 1. We call the Generalized Hypergeometric Function, otherwise known as the Generalized Gauss function, the series

$$ {}_pF_q[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z] = \sum_{n=0}^{\inf} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n}{(b_1)_n(b_2)_n \ldots (b_q)_n} \frac{z^n}{n!} \qquad (1) $$

where a_1, $a_2,\ldots,$ a_p and b_1, $b_2,\ldots,b_q$ are complex parameters, z is a complex variable. We denote:

$$ (a)_n = a(a+1) \ldots (a+n-1) \qquad (2) $$

We also denote the above series as ${}_pF_q[a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z]$ or ${}_pF_q[(a);(b);z]$ or simply ${}_pF_q(z)$.

The key ideas in our design, depicted in figure 1, are:

Stage 1. Represent the Special Functions, if possible, as particular instances of the Generalized Hypergeometric Function.

Stage 2. Provide a fairly general formula to integrate the results of stage 1.

Stage 3. Take the result of stage 2 involving a Generalized Hypergeometric Function, and reduce it to an elementary and/or Special Function(s).

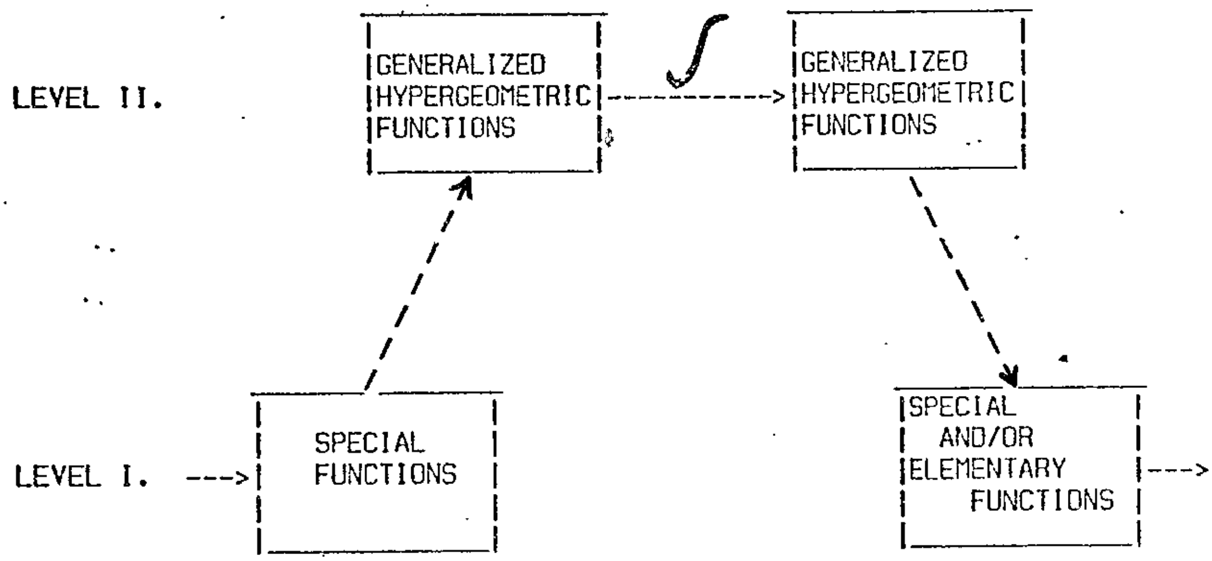

Figure 1.

Hence, our design alternates between two levels:

Level 1. The expression involves Special and/or Elementary Functions.

Level 2. The expression involves Generalized Hypergeometric Functions.

We will give next a simple illustration of the above scheme, but first let's provide one more definition.

Definition 2. We call the Laplace Transform of a real or complex function f(t), defined for all real nonnegative values of t, the integral

$$\int_0^\infty f(t)e^{-pt}\,dt \tag{3}$$

if it exists for some values of the complex variable p. It is written L[f(t)] and determines a function F(p), thus

$$L[f(t)] = \int_0^\infty f(t)e^{-pt}\,dt = F(p) \tag{4}$$

We next proceed with a simple illustration of this approach.

Given input

$$t^{-3/2}\, I_3(2a^{1/2}t^{1/2})\, e^{-pt} \tag{5}$$

where I_3 is a modified Bessel function of the first kind [4], [13], the following will take place in each of the three stages:

Stage 1.

Since

$$I_v(z) = e^{-v\pi i/2}\, J_v(ze^{\pi i/2}) \tag{6}$$

expression (5) becomes

$$i t^{-3/2}\, J_3(2ia^{1/2}t^{1/2})\, e^{-pt} \tag{7}$$

Since

$$J_v(z) = \frac{z^v}{2^v\Gamma(v+1)}\; {}_0F_1[\; v+1;\; -1/4\; z^2] \tag{8}$$

(7) becomes

$$\frac{a^{3/2}}{6p}\ {}_0F_1[\ 4;\ at]\ e^{-pt} \tag{9}$$

Stage 2.

In this stage we recognize that our input is a Laplace transformable expression. Hence, we integrate by using the following formula [11].

$$\int_0^\infty t^{s-1}\,{}_mF_n(a_1,\ldots,a_m;r_1,\ldots,r_n;(lt)^k)e^{-pt}\,dt \tag{10}$$

$$= \Gamma(s)p^{-s}\,{}_{m+k}F_n(a_1,\ldots,a_m,\frac{s}{k},\frac{s+1}{k},\ldots,\frac{s+k-1}{k};r_1,r_2,\ldots,r_n;(\frac{kl}{p})^k)$$

which is valid for Re(s) > 0, m+k < n+1, where k,m,n are integers.

Thus (9) becomes

$$\frac{a^{3/2}}{6p}\ {}_1F_1[\ 1;\ 4;\ a/p] \tag{11}$$

Stage 3.

At stage 3, we apply to (11) the following "Kummer's transformation" [3]

$${}_1F_1[\ a;\ r;\ z] = e^z\ {}_1F_1[\ r-a;\ r;\ -z] \tag{12}$$

and (11) reduces to

$$\frac{a^{3/2}}{6p}\ e^{a/p}\ {}_1F_1[\ 3;\ 4;\ -a/p] \tag{13}$$

We recognize that the series in (13) is an instance of an Incomplete Gamma function [3], because

$${}_1F_1[\ a;\ a+1;\ -x] = ax^{-a}\,\gamma(a,x) \tag{14}$$

Therefore, (14) finally becomes

$$\frac{e^{a/p}\, p^2}{2a^{3/2}} \gamma(3, a/p) \qquad (15)$$

Hence, our research was split into as many parts as there are stages in the above illustration. As it turned out decisions on designs of stages one and two are somewhat interdependent, whereas stage three is totally independent. Of these three stages, the third stage give rise to the most serious difficulties. Thus our attention and emphasis was shifted most of the time to problems of this third stage. As a consequence, Chapter 3 that refers to the reduction methods occupies the focus of this thesis. Chapter 2 describes the two earlier stages.

Chapter 3 concetrates on two groups of reduction methods of the Generalized Hypergeometric Function:

1. Those that are dependent on the number of parameters (we call them "general reduction methods").

2. Those that are independent of the number of parameters (we call them "special reduction methods").

From the second group we have been principally concerned with reduction procedures of the following instances of the Generalized Hypergeometric Function: ${}_0F_0(z)$, which is actually the exponential function; ${}_0F_1(z)$, which mainly involves the Bessel functions; ${}_1F_0(z)$, which includes the binomial functions; ${}_1F_1(z)$, the so called "Confluent Hypergeometric Functions"; ${}_2F_1(z)$, the (Gauss)

Hypergeometric Functions which include the Elementary functions in addition to some Special Functions.

Chapter 2 begins with a short overview of the mathematical background. The reader might want to consult the references. However, we feel that this is not necessary to capture the points of this thesis. Chapter 2 next demonstrates the policy we adopted for each of the approximately fifty Special Functions so that our goal, viewing each Special Function as a Generalized Hypergeometric Function whenever possible, can be accomplished without too much difficulty in the integration stage, difficulty that could have been caused as a result of generalizing the problem. Finally, Chapter 2 ends with the integration stage. Here, we indicate our major design decisions for the table look up in terms of lemmas. These lemmas help us to keep the number of formulas in our table down to a minimum. Moreover, we further ease the burden of the table for composite function cases (rather "extraneous cases") by appropriately utilizing different integral transform properties that recursively call our scheme as illustrated in this chapter (fig. 1) for relatively simpler cases.

Chapter 2

SPECIAL AND GENERALIZED FUNCTIONS. STAGES I - II.

The Generalized Hypergeometric Function has been defined as a series (Chapter 1). This series satisfies the differential equation:

$$\left(z\frac{d}{dz}\left(z\frac{d}{dz}+b_1-1\right)\left(z\frac{d}{dz}+b_2-1\right)\ldots\left(z\frac{d}{dz}+b_q-1\right) - z\left(z\frac{d}{dz}+a_1\right)\left(z\frac{d}{dz}+a_2\right)\ldots\left(z\frac{d}{dz}+a_p\right)\right)y=0 \qquad (1)$$

The series ${}_pF_q(z)$ converges under the following conditions:

1. For all values of z, real or complex as long as $p \leq q$
2. For all values of z such that $|z| < 1$, as long as $p = q+1$
3. For $z = 1$ if $\text{Real}\left(\sum_{v=1}^{q} b_v - \sum_{v=1}^{p} a_v\right) > 0$
4. For $z = -1$ if $\text{Real}\left(\sum_{v=1}^{q} b_v - \sum_{v=1}^{p} a_v\right) > -1$

In case that $p > q+1$, the series never converges, except when $z = 0$, while the function is only defined when the series terminates and this happens when at least one of the parameters in the L_1 list is zero or some negative integer (see Chapter 3).

An obvious conclusion of Definition 1, of the Generalized Hypergeometric Function is that any permutation of the members of L_1 or L_2 lists does not affect the Generalized Function.

A notion which is particularly useful for our reduction purposes is "contiguity".

<u>Definition 1.</u> Two Generalized Hypergeometric Functions

$$_pF_q[\ L_1;\ L_2;\ z] \quad \text{and} \quad _pF_q[\ L_1';\ L_2';\ z] \tag{2}$$

are called contiguous if they are alike except for one pair of parameters in which they differ by a unity.

Every Generalized Hypergeometric Function $_pF_q(z)$ is contiguous to $2p+2q$ others. Hence, the Hypergeometric Function $_2F_1[\ a,\ b;\ c;\ z]$ is contiguous to $_2F_1[\ a+1,\ b;\ c;\ z]$ and obviously to five others.

If we use the following abbreviations:

$$F \;=\; F[\ a_1,\ a_2,\ \ldots\ ,\ a_p;\ b_1,\ b_2,\ \ldots\ ,\ b_q;\ z] \tag{3}$$

$$F[a_1\pm1] \;=\; F[\ a_1\pm1,\ a_2,\ \ldots\ ,\ a_p;\ b_1,\ \ldots\ ,\ b_q;\ z] \tag{4}$$

$$F[b_1\pm1] \;=\; F[\ a_1,\ a_2,\ \ldots\ ,\ a_p;\ b_1\pm1,\ \ldots\ ,\ b_q;\ z] \tag{5}$$

then we have the following contiguous relations presented in tables one and two for the Gauss Hypergeometric Functions and the Confluent Hypergeometric Functions correspondingly.

$$\{b_1-2a_1+(a_1-a_2)z\}F + a_1(1-z)F[a_1+1] = (b_1-a_1)F[a_1-1] \tag{6}$$

$$(a_2-a_1)F + a_1F[a_1+1] = a_2F[a_2+1] \tag{7}$$

$$(b_1-a_1-a_2)F + a_1(1-z)F[a_1+1] = (b_1-a_2)F[a_2-1] \tag{8}$$

$$b_1\{a_1+(a_2-b_1)z\}F + (b_1-a_1)(b_1-a_2)zF[b_1+1] = a_1b_1(1-z)F[a_1+1] \tag{9}$$

$$(b_1-a_1-1)F + a_1F[a_1+1] = (b_1-1)F[b_1-1] \tag{10}$$

$$(b_1-a_1-a_2)F + a_2(1-z)F[a_2+1] = (b_1-a_1)F[a_1-1] \tag{11}$$

$$(a_2-a_1)(1-z)F + (b_1-a_2)F[a_2-1] = (b_1-a_1)F[a_1-1] \tag{12}$$

$$b_1(1-z)F + (b_1-a_2)zF[b_1+1] = b_1F[a_1-1] \tag{13}$$

$$\{a_1-1+(1+a_2-b_1)z\}F + (b_1-a_1)F[a_1-1] = (b_1-1)(1-z)F[b_1-1] \tag{14}$$

$$\{b_1-2a_2+(a_2-a_1)z\}F + a_2(1-z)F[a_2+1] = (b_1-a_2)F[a_2-1] \tag{15}$$

$$(b_1-a_2-1)F + a_2F[a_2+1] = (b_1-1)F[b_1-1] \tag{16}$$

$$b_1(1-z)F + (b_1-a_1)zF[b_1+1] = b_1F[b_1-1] \tag{17}$$

$$\{a_2-1+(1+a_1-b_1)z\}F + (b_1-a_2)F[a_2-1] = (b_1-1)(1-z)F[b_1-1] \tag{18}$$

$$b_1\{b_1-1+(1+a_1+a_2-2b_1)z\}F + (b_1-a_1)(b_1-a_2)zF[b_1+1] = b_1(b_1-1)(1-z)F[b_1-1] \tag{19}$$

Table 1.

$$(b_1-a_1)F[a_1-1] + (2a_1-b_1+z)F - a_1F[a_1+1] = 0 \tag{20}$$

$$b_1(b_1-1)F[b_1-1] - b_1(b_1-1+z)F + (b_1-a_1)zF[b_1+1] = 0 \tag{21}$$

$$(a_1-b_1+1)F - a_1F[a_1+1] + (b_1-1)F[b_1-1] = 0 \tag{22}$$

$$b_1F - b_1F[a_1-1] - zF[b_1+1] = 0 \tag{23}$$

$$b_1(a_1+z)F - (b_1-a_1)zF[b_1+1] - a_1b_1F[a_1+1] = 0 \tag{24}$$

$$(a_1-1+z)F + (b_1-a_1)F[a_1-1] - (b_1-1)F[b_1-1] = 0 \tag{25}$$

Table 2.

Generally, such relations exist for any Generalized Hypergeometric Function [14].

Of great interest to us are also different transformations of the Confluent Hypergeometric Functions as well as of the Gauss Hypergeometric Functions. For example, linear transformations are applicable to both Confluent and Gauss Hypergeometric Functions while quadratic, cubic and other of higher order transformations are available for the Gauss Hypergeometrics. Analytic tables for the quadratic transformations and some important qubic ones are provided in the Appendix 2. For more information concerning the above transformations the reader should consult Goursat's paper [15].

The following differential relations depicted in tables three and four for the Confluent and Gauss Hypergeometric Functions correspondingly also hold and are of great interest to us [3].

$$\frac{d^n}{dz^n}\, {}_2F_1[\, a,\ b;\ c;\ z] = \frac{(a)_n (b)_n}{(c)_n}\, {}_2F_1[\, a+n,\ b+n;\ c+n;\ z] \tag{26}$$

$$(a)_n\, z^{a-1}\, {}_2F_1[\, a+n,\ b;\ c;\ z] = \frac{d^n}{dz^n} [z^{a+n-1}\, {}_2F_1[\, a,\ b;\ c;\ z]] \tag{27}$$

$$(c-n)_n\, z^{c-1-n}\, {}_2F_1[\, a,\ b;\ c-n;\ z] = \frac{d^n}{dz^n}[z^{c-1}\, {}_2F_1[\, a,\ b;\ c;\ z]] \tag{28}$$

$$(c-a)_n\, z^{c-a-1}(1-z)^{a+b-c-n}\, {}_2F_1[\, a-n,\ b;\ c;\ z] \tag{29}$$

$$= \frac{d^n}{dz^n} [z^{c-a+n-1}(1-z)^{a+b-c}\, {}_2F_1[\, a,\ b;\ c;\ z]]$$

$$\frac{(c-a)_n(c-b)_n}{(c)_n}(1-z)^{a+b-c-n}\,{}_2F_1[\,a,\ b;\ c+n;\ z] = \frac{d^n}{dz^n}[(1-z)^{a+b-c}\,{}_2F_1[\,a,\ b;\ c;\ z]] \tag{30}$$

$$\frac{(-1)_n(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}\,{}_2F_1[\,a+n,\ b;\ c+n;\ z] = \frac{d^n}{dz^n}[(1-z)^{a+n-1}\,{}_2F_1[\,a,\ b;\ c;\ z]] \tag{31}$$

$$(c-n)_n z^{c-1-n}(1-z)^{b-c}\,{}_2F_1[\,a-n,\ b;\ c-n;\ z] = \frac{d^n}{dz^n}[z^{c-1}(1-z)^{b-c+n}\,{}_2F_1[\,a,\ b;\ c;\ z]] \tag{32}$$

$$(c-n)_n z^{c-1-n}(1-z)^{a+b-c-n}\,{}_2F_1[\,a-n,\ b-n;\ c-n;\ z] = \frac{d^n}{dz^n}[z^{c-1}(1-z)^{a+b-c}\,{}_2F_1[\,a,\ b;\ c;\ z]] \tag{33}$$

Table 3.

$$\frac{d^n}{dz^n}\,{}_1F_1[\,a;\ c;\ z] = \frac{(a)_n}{(c)_n}\,{}_1F_1[\,a+n;\ c+n;\ z] \tag{34}$$

$$\frac{d^n}{dz^n}[z^{a+n-1}\,{}_1F_1[\,a;\ c;\ z]] = (a)_n z^{a-1}\,{}_1F_1[\,a+n;\ c;\ z] \tag{35}$$

$$\frac{d^n}{dz^n}[z^{c-1}\,{}_1F_1[\,a;\ c;\ z]] = (-1)^n(1-c)_n z^{c-1-n}\,{}_1F_1[\,a;\ c-n;\ z] \tag{36}$$

$$\frac{d^n}{dz^n}[e^{-z}\,{}_1F_1[\,a;\ c;\ z]] = (-1)^n\frac{(c-a)_n}{(c)_n}e^{-z}\,{}_1F_1[\,a;\ c+n;\ z] \tag{37}$$

$$\frac{d^n}{dz^n}[e^{-z}z^{c-a+n-1}\ {}_1F_1[\ a;\ c;\ z]] \qquad (38)$$

$$= (c-a)_n e^{-z} z^{c-a-1}\ {}_1F_1[\ a-n;\ c;\ z]$$

Table 4.

Most of the Special Functions (eg. Bessel, Legendre, Whittaker etc.) are solutions of a particular instance of the differential equation (1). They have some series expansions (instance of the Definition 1 of Chapter 1), they satisfy properties such as contiguity etc. [3], [12]. Hence, we will concentrate on the different relations essential to our design and ignore definitions and comments on every single Special Function, unless really necessary. For an extensive study of the different Special Functions the reader should consult the Bateman Manuscript Project [3].

2.1. THE FIRST STAGE

Our main concern during stage one of our definite integration scheme is

1. To represent the Special Functions of the given expression as a Generalized Hypergeometric Function

2. To be sure that the resulting expression has the appropriate format to be successfully proccessed in stage two.

Hence, we will be first concerned with the representation of Special Functions in terms of the Generalized Hypergeometric Function.

As we have mentioned, we have dealt with approximately fifty Special Functions. We have divided the set of the Special Functions into two major types. The first type includes all Special Functions that are directly transformed through some relation into a Generalized Hypergeometric Function, and the second type includes those that are expressed in terms of other Special Functions and ultimately are expressed in terms of Special Functions of the first type. This classification has been influenced by the tendency to utilize and manipulate as few Special Functions as is necessary.

Let us start the discussion with the class of Bessel functions.

The Bessel function of the first kind $J_v(z)$ is of the first type and is automatically transformed into a Generalized Hypergeometric Function through the relation

$$J_v(z) = \frac{z^v}{\Gamma(v+1)} {}_0F_1[\ v+1;\ -\frac{z^2}{4}] \tag{39}$$

The Modified Bessel function of the first kind $I_v(z)$ is of the second type. It is transformed into a Bessel function of the first kind through the relation

$$I_v(z) = e^{-v\pi i/2} J_v(ze^{\pi i/2}) \tag{40}$$

where $J_v(z)$ is of the first type.

The Bessel function of the second kind $Y_v(z)$ is a member of the set of functions of the second type, for noninteger values of the index v because

$$Y_v(z) = (\cos(v\pi) J_v(z) - J_{-v}(z)) \csc(v\pi) \tag{41}$$

where relation (41) holds for noninteger values of the index v and where $J_v(z)$ is of the first type. In case that v has an integer value, the following relations hold for our $Y_v(z)$

$$Y_n(z) = \lim_{v \to n} Y_v(z) \tag{42}$$

$$Y_{-n}(z) = (-1)^n Y_n(z), \quad n \in N \tag{43}$$

It can be shown that relations (42) and (43) imply that $Y_n(z)$ for $n \in Z$ is not a function of type two (and hence certainly not of type one). Thus, $Y_v(z)$ for $v \in Z$ gives rise to some complications in our scheme. The case $Y_n(z)$, $n \in Z$

has to be handled individually in stage one as well as in stage two. For an extensive analysis see Watson and Tranter [4], [13].

The Modified Bessel function of the second kind $K_v(z)$ belongs to the set of second type functions for noninteger values of the index v because

$$K_v(z) = 1/2 \pi \csc(\pi v) (I_{-v}(z) - I_v(z)) \qquad (44)$$

holds for noninteger values of the index v where $I_v(z)$ is a type one function. For $v \in Z$ we have

$$K_{-n}(z) = (-1)^n K_n(z) \qquad (45)$$

$$K_n(z) = \lim_{v \to n} K_v(z), \quad n \in N \qquad (46)$$

Thus the Modified Bessel function of the second kind $K_v(z)$ is handled in the same fashion as $Y_v(z)$ function is handled.

The first kind of Hankel function $H_{v,1}(z)$ (also called the first kind of the third kind Bessel function), belongs to the second type of functions and can be obtained from the following relation

$$H_{v,1}(z) = J_v(z) + iY_v(z) \qquad (47)$$

where $J_v(z)$ is a first type function and $Y_v(z)$ a second type one. Obviously, no special handling for the function $H_{v,1}(z)$ for $v \in Z$ is required because $Y_v(z)$ function takes care of that.

In a similar manner the second kind Hankel function $H_{v,2}(z)$, is a second type function and can be obtained from the relation

$$H_{v,2}(z) = J_v(z) - iY_v(z) \qquad (48)$$

Let us consider next the Kelvin functions and those related to them. They are all ultimately expressible in terms of Bessel and Modified Bessel functions as shown by relations (49) through (52):

$$\mathrm{ber}_v(z) = 1/2\ J_v(z\ e^{3\pi i/4}) + 1/2\ J_v(z\ e^{-3\pi i/4}) \tag{49}$$

$$\mathrm{bei}_v(z) = i/2\ J_v(z\ e^{-3\pi i/4}) - i/2\ J_v(z\ e^{3\pi i/4}) \tag{50}$$

$$\mathrm{ker}_v(z) = 1/2\ K_v(z\ e^{\pi i/4}) + 1/2\ K_v(z\ e^{-\pi i/4}) \tag{51}$$

$$\mathrm{kei}_v(z) = i/2\ K_v(z\ e^{-\pi i/4}) - i/2\ K_v(z\ e^{\pi i/4}) \tag{52}$$

and therefore they belong to our set of functions of the second type. Particularly, here we have

$$\mathrm{ber}(z) = \mathrm{ber}_0(z) \qquad \mathrm{bei}(z) = \mathrm{bei}_0(z) \tag{53}$$

$$\mathrm{ker}(z) = \mathrm{ker}_0(z) \qquad \mathrm{kei}(z) = \mathrm{kei}_0(z) \tag{54}$$

Like the Kelvin functions, the Airy functions Ai(t) and Bi(t) are ultimately expressible in terms of Bessel functions as the following relations show:

$$\begin{aligned} \mathrm{Ai}(t) = {} & 1/3\ t^{1/2}\ e^{\pi i/6}\ J_{-1/3}(2it^{3/2}/3) \\ & - 1/3\ t^{1/2}\ e^{-\pi i/6}\ J_{1/3}(2it^{3/2}/3) \end{aligned} \tag{55}$$

$$\begin{aligned} \mathrm{Bi}(t) = {} & (t/3)^{1/2}\ e^{\pi i/6}\ J_{-1/3}(2it^{3/2}/3) \\ & + (t/3)^{1/2}\ e^{-\pi i/6}\ J_{1/3}(2it^{3/2}/3) \end{aligned} \tag{56}$$

thus they are also second type functions.

The Lommel function $s_{\mu,v}(z)$ is of the first type as the following relation show

$$s_{\mu,v}(z) = \frac{z^{\mu+1}}{(\mu-v+1)(\mu+v+1)}\, {}_1F_2[\,1;\ \frac{\mu-v+3}{2},\ \frac{\mu+v+3}{2};\ -\frac{z^2}{4}] \qquad (57)$$

The Lommel function $S_{\mu,v}(z)$ is of the second type because of the following relation

$$S_{\mu,v}(z) = s_{\mu,v}(z) \qquad (58)$$
$$+\ 2^{\mu-1}\,\Gamma(\frac{\mu-v+1}{2})\,\Gamma(\frac{\mu+v+1}{2})\,(\sin(\frac{\mu-v}{2}\pi)\,J_v(z) - \cos(\frac{\mu-v}{2}\pi)\,Y_v(z))$$

where $s_{\mu,v}(z)$ and $J_v(z)$ are of the first type while $Y_v(z)$ is of the second type, requiring for $v \in Z$ some special treatment (as indicated before). Furthermore, relation (58) holds provided that $(\mu-v+1)/2$, $(\mu+v+1)/2$ are nonnegative integers.

The Struve function $H_v(z)$ is a second type function, since

$$H_v(z) = 2^{1-v}\,\pi^{-1/2}\,(\Gamma(v+1/2))^{-1}\,s_{v,v}(z) \qquad (59)$$

Likewise, the Struve function $L_v(z)$ is of the second type because

$$L_v(z) = e^{-(v+1)\pi i/2}\,H_v(z\,e^{i\pi/2}) \qquad (60)$$

After the Bessel family of functions we come to the Gauss Hypergeometric Functions.

The Legendre functions $P_{v,\mu}(z)$ and $Q_{v,\mu}(z)$ are both first type functions since

$$P_{v,\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1[\,-v,\ v+1;\ 1-\mu;\ 1/2-z/2] \qquad (61)$$

$$Q_{v,\mu}(z) = \frac{e^{\mu\pi i}\pi^{1/2}\Gamma(\mu+v+1)}{2^{v+1}\Gamma(v+3/2)} z^{-\mu-v-1} (z^2-1)^{\mu/2} \qquad (62)$$

$${}_2F_1\left[\frac{\mu+v+1}{2},\ \frac{\mu+v+2}{2};\ v\pm\frac{3}{2};\ z^{-2}\right]$$

Particularly, here the following relations hold

$$P_{v,0}(z) = P_v(z) \qquad\qquad Q_{v,0}(z) = Q_v(z) \qquad (63)$$

furthermore, for $\mu = 0$ and $v = n = 0, 1, 2, \ldots$ we get the Legendre Polynomials.

The (Complete) Elliptic integrals (functions) are also type one functions and are given by the following relations

$$K(k) = \pi/2\ {}_2F_1[\,1/2,\ 1/2;\ 1;\ k^2] \qquad (64)$$

$$E(k) = \pi/2\ {}_2F_1[\,-1/2,\ 1/2;\ 1;\ k^2] \qquad (65)$$

The Orthogonal Polynomials of Jacobi are of the first type and are given by the following relation

$$P_{n,(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1[\,-n,\ n+\alpha+\beta+1;\ \alpha+1;\ (1-x)/2] \qquad (66)$$

The Orthogonal Polynomials of Gegenbauer $C_{n,v}(x)$, Legendre $P_n(x)$ and Tchebichef $T_n(x)$ and $U_n(x)$ are of the second type given by the following relations

$$C_{n,v}(x) = \frac{(2v)_n}{(v+1/2)_n} P_{n,(v-1/2,v-1/2)}(x) \tag{67}$$

$$P_n(x) = C_{n,1/2}(x) \tag{68}$$

$$T_n(x) = n/2\, C_{n,0}(x) \tag{69}$$

$$U_n(x) = C_{n,1}(x) \tag{70}$$

We next consider the Confluent Hypergeometric Functions [16]. The Whittaker function $M_{\kappa,\mu}(z)$ which covers the whole spectrum of the Confluent Hypergeometric Functions, is a type one function and is given by the relation

$$M_{\kappa,\mu}(z) = z^{1/2+\mu}\, e^{-z/2}\, {}_1F_1[\, 1/2+\mu-\kappa;\ 2\mu+1;\ z] \tag{71}$$

The Incomplete Gamma function $\gamma(\alpha,x)$ is also a type one function and is given by

$$\gamma(\alpha,x) = \alpha^{-1}x^{\alpha}\, {}_1F_1[\, \alpha;\ \alpha+1;\ -x] \tag{72}$$

The second Whittaker function $W_{\kappa,\mu}(z)$ is a type two function and given by the relation

$$W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\kappa)} M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\kappa)} M_{\kappa,-\mu}(z) \tag{73}$$

as long as μ does not take integer values and the quantities $1/2-\mu-\kappa$ and $1/2+\mu-\kappa$ are not negative numbers or zero. Otherwise, $W_{\kappa,\mu}(z)$ is considered separately in both stages one and two.

The Parabolic Cylinder function is a type two and is given by

$$D_v(z) = 2^{v/2+1/4} z^{-1/2} W_{v/2+1/4,1/4}(z^2/2) \qquad (74)$$

for $v = n = 0, 1, 2, \ldots$ we have the Parabolic Cylinder Polynomials.

The Bateman function $k_v(z)$, the two Error functions $Erf(x)$ and $Erfc(x)$, the Incomplete Gamma function $\Gamma(\alpha,x)$, the exponential integral and related functions $Ei(x)$, $si(x)$, $Si(x)$ and $Ci(x)$ are all type two functions and are given by the following relations:

$$k_{2v}(z) = \Gamma(v+1)^{-1} W_{v,1/2}(2z) \qquad (75)$$

$$Erf(x) = 1/2\ \gamma(1/2,x^2) \qquad (76)$$

$$Erfc(x) = (\pi x)^{-1/2} e^{-x^2/2} W_{-1/4,1/4}(x^2) \qquad (77)$$

$$\Gamma(\alpha,x) = x^{(\alpha-1)/2} e^{-x/2} W_{(a-1)/2,a/2}(x) \qquad (78)$$

$$-Ei(-x) = e^{-x/2} x^{-1/2} W_{-1/2,0}(x) \qquad (79)$$

$$Si(x) = \pi/2 - 1/2\ i^{1/2} e^{-ix/2} x^{-1/2} W_{-1/2,0}(ix) + 1/2\ i^{3/2} e^{ix/2} x^{-1/2} W_{-1/2,0}(ix) \qquad (80)$$

$$Ci(x) = -1/2\ i^{3/2} e^{-ix/2} x^{-1/2} W_{-1/2,0}(ix) - 1/2\ i^{1/2} e^{ix/2} x^{-1/2} W_{-1/2,0}(-ix) \qquad (81)$$

$$si(x) = -\pi/2 + Si(x) \qquad (82)$$

Laguerre Polynomials have been assigned to the first type set and are given by the following relation

$$L_{n,\alpha}(x) = \binom{n+\alpha}{n} {}_1F_1[-n;\ \alpha+1;\ x] \qquad (83)$$

The Orthogonal Polynomials of Hermite, special cases of the Parabolic Cylinder functions, belong to the second type set and are given by the following relation

$$He_n(x) = e^{x^2/4} D_n(x) \tag{84}$$

In a similar way we have considered products of Special Functions which can be expressed as a single Generalized Hypergeometric Function. Thus the product of two Bessel functions $J_v(z)J_\mu(z)$ is of the first type and is transformed into a Generalized Hypergeometric Function through the relations (85) and (39)

$$
\begin{aligned}
&{}_0F_1[\,\rho;\ z]\,{}_0F_1[\,\sigma;\ z] \\
&\qquad = {}_2F_3[\,\rho/2+\sigma/2,\ \rho/2+\sigma/2-1/2;\ \rho,\ \sigma,\ \rho+\sigma-1;\ 4z]
\end{aligned}
\tag{85}
$$

On the other hand, the product $I_v(z)K_\mu(z)$, where $I_v(z)$, $K_\mu(z)$ are modified Bessel functions of the first and second kind respectively, belongs to the second type and is ultimately expressible in terms of functions of the first type, for noninteger values of the index μ. Similar arguments are applicable to similar products of the so far mentioned Special Functions. Table 5 provides products of Generalized Hypergeometric Functions which are expressible in terms of one Generalized Hypergeometric Function.

$$
\begin{aligned}
&({}_2F_1[\,a,\ b;\ a+b+1/2;\ z])^2 \\
&\qquad = {}_3F_2[\,2a,\ a+b,\ 2b;\ a+b+1/2,\ a+2b;\ z]
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
&{}_0F_1[\,\rho;\ z]\,{}_0F_1[\,\rho;\ -z] \\
&\qquad = {}_0F_3[\,\rho,\ \rho/2,\ \rho/2+1/2;\ -z^2/4]
\end{aligned}
\tag{87}
$$

$$ {}_1F_1[\ \alpha;\ \rho;\ z]\,{}_1F_1[\ \alpha;\ \rho;\ -z] = {}_2F_3[\ \alpha,\ \rho-\alpha;\ \rho,\ \rho/2,\ (\rho+1)/2;\ z^2/4] \qquad (88) $$

$$ {}_1F_1[\ \alpha;\ 2\alpha;\ z]\,{}_1F_1[\ \beta;\ 2\beta;\ -z] = {}_2F_3[\ (\alpha+\beta)/2,\ (\alpha+\beta+1)/2;\ \alpha+1/2,\ \beta+1/2,\ \alpha+\beta;\ z^2/4] \qquad (89) $$

$$ {}_2F_1[\ \alpha,\ \beta;\ \alpha+\beta-1/2;\ z]\,{}_2F_1[\ \alpha,\ \beta;\ \alpha+\beta+1/2;\ z] = {}_3F_2[\ 2\alpha,\ 2\beta;\ \alpha+\beta;\ 2\alpha+2\beta-1,\ \alpha+\beta+1/2;\ z] \qquad (90) $$

$$ {}_2F_1[\ \alpha,\ \beta;\ \alpha+\beta-1/2;\ z]\,{}_2F_1[\ \alpha-1,\ \beta;\ \alpha+\beta-1/2;\ z] = {}_3F_2[\ 2\alpha-1,\ 2\beta,\ \alpha+\beta-1;\ 2\alpha+2\beta-2,\ \alpha+\beta-1/2;\ z] \qquad (91) $$

Table 5.

Due to the importance of the area of "products of Special Functions" in our scheme, we plan in the near future to further investigate this area.

Relations (39) through (91) actually show the way we have chosen to transform a given Special Function to its corresponding Generalized Hypergeometric representation, whenever this is possible. As it has been already indicated, we may need many intermediate steps. For example, the Orthogonal Polynomial of Legendre is first transformed into a Tchebychef which is next transformed into a Jacobi and at last to a Hypergeometric form. An obvious question that arises is, "why do we need the "intermediate steps"? Why do we not transform the Special Function in one step to its Generalized Hypergeometric representation? For example, why is the Legendre Polynomial not given immediately by relation (92) ?

$$ P_n(x) = {}_2F_1[\ -n,\ n+1;\ 1;\ 1/2-x/2] \qquad (92) $$

The reason that we decided to follow this very "conservative" strategy is as follows:

> At every level of generalization some particular properties and knowledge might be applicable, and yet is not generalizable and therefore not applicable in a higher level.

The history of Special Functions reveals that people have always come up with "new problems" requiring definition of "new Special Functions" which generally have some "tight" or "loose" connection (and which generally add new chaos to the already existing one) with the set of "fifty" we selected to work with. Hence, we feel that our strategy would best serve further possible additions of new Special Functions to the already existing set.

To give support to our argument, we give as an example relations (93) and (94) which hold only for Bessel functions and some Orthogonal Polynomialss respectively.

$$Y_{-n}(z) = (-1)^n Y_n(z), \quad n \in N \tag{93}$$

$$POLY_n(-x) = (-1)^n POLY_n(x), \quad n \in N \tag{94}$$

where $POLY_n(x)$ is the Orthogonal Polynomial of Legendre or Tchebychef or Gegenbauer or Jacobi, etc.

In the light of these remarks, it is seen that the aforementioned strategy also facilitates attainment of our second objective (mentioned at the begining of this section) and provides more flexibility in interfacing stages one and two. For the time being, let us investigate the second stage and see its problems. From time to time we will return to stage one and resolve interfacing problems of stages one and two.

2.2. THE SECOND STAGE

In the integration stage we determine if an expression is Laplace transformable or Fourier transformable or any other kind of transformable expression.

We will now add to the already mentioned definition of Laplace Transforms some definitions of other Integral transforms.

Definition 2. We define the Fourier cosine, sine and exponential transforms to be the following corresponding integrals:

$$\int_0^\infty f(x)\cos(xy)\,dx \tag{95}$$

$$\int_0^\infty f(x)\sin(xy)\,dx \tag{96}$$

$$\int_0^\infty f(x)e^{-ixy}\,dx \tag{97}$$

Definition 3. We call the Hankel transform of order v of the function f(x) the integral

$$\int_0^\infty f(x)J_v(xy)(xy)^{1/2}\,dx \tag{98}$$

where y is a positive real variable.

Definition 4. We call the Y-transform of order v of the function f(x) the integral

$$\int_0^\infty f(x)Y_v(xy)(xy)^{1/2}\,dx \tag{99}$$

where y is assumed a positive real variable.

Definition 5. We call the K-transform of order v of the function f(x) the integral

$$\int_c^{\infty} f(x) K_v(xy)(xy)^{1/2}\, dx \tag{100}$$

where y is regarded as a complex variable.

Definition 6. We call the Stieltjes-transform of f(x) the integral

$$\int_o^{\infty} f(x)(x+y)^{-1}\, dx \tag{101}$$

where the integration is over the positive real x-axis, and y is a complex variable ranging over the complex y-plane cut along the negative real axis.

We next give some key remarks in terms of lemmas

Lemma 1. The Hankel transform of a function f(x) reduces to the Fourier sine transform for v = 1/2. Particularly, the following relation holds

$$\int_o^{\infty} f(x) J_{1/2}(xy)(xy)^{1/2}\, dx = (2/\pi)^{1/2} \int_o^{\infty} f(x) \sin(xy)\, dx \tag{102}$$

Proof This is immediate from definitions (96) and (98).

Similarly, we have.

Lemma 2. The Hankel transform of a function f(x) reduces to the Fourier cosine transform for v = -1/2. Particularly, the following relation holds

$$\int_o^{\infty} f(x) J_{-1/2}(xy)(xy)^{1/2}\, dx = (2/\pi)^{1/2} \int_o^{\infty} f(x) \cos(xy)\, dx \tag{103}$$

Lemma 3. For the exponential Fourier transform the following relation holds

$$\int_0^\infty f(x)e^{-ixy}\,dx \tag{104}$$
$$= \int_0^\infty [f(x)+f(-x)]\cos(xy)\,dx \;-\; i\int_0^\infty [f(x)-f(-x)]\sin(xy)\,dx$$

<u>Proof</u> This is an immediate consequence of relations (95), (96) and (97).

<u>Lemma 4.</u> The K-transform of order v of a function f(x) reduces to the Laplace transforms of that function for $v = \pm 1/2$. Particularly,

$$(\pi/2)^{1/2} \int_0^\infty f(x)e^{-xy}\,dx \tag{105}$$
$$= \int_0^\infty f(x)K_{1/2}(xy)(xy)^{1/2}\,dx$$
$$= \int_0^\infty f(x)K_{-1/2}(xy)(xy)^{1/2}\,dx$$

<u>Proof</u> It is immediate if we notice that

$$K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \tag{106}$$

As we saw in a previous paragraph, we divided the set of Special Functions into two major types. In particular we saw that most kinds of Bessel functions, were expressible in terms of the Bessel function J_v except for some indices. A natural consequence is that here the K and Y transforms can be expressed in terms of the Hankel transforms for certain indices. Particularly, we have.

<u>Lemma 5.</u> The following relation holds as long as the index v is not an integer.

$$\int_0^\infty f(x)Y_v(xy)(xy)^{1/2}\,dx \tag{107}$$
$$= \cot(v\pi) \int_0^\infty f(x)J_v(xy)(xy)^{1/2}\,dx$$
$$- \csc(v\pi) \int_0^\infty f(x)J_v(xy)(xy)^{1/2}\,dx$$

Proof This is an immediate consequence of relation (41) of this Chapter.

A similar relation to relation (107) can be also established between Hankel and K transforms. However, it is of very limited practical value since the Hankel transforms for y complex very rarely converge.

Lemma 6. The Stieltjes transforms of a function f(x) where x lies on the positive real axis and y complex, are iterated Laplace Transforms. Namely,

$$\int_0^\infty f(x)\,(x+y)^{-1}\,dx = \int_0^\infty \Big[\int_0^\infty f(x)\,e^{-xt}\,dt\Big]\,e^{-xy}\,dy \tag{108}$$

The above mentioned relationships among the different transforms are not the only ones. Hence, the following relationships also exist between the Laplace and K transforms.

$$\int_0^\infty f(x)K_v(xy)\,(xy)^{1/2}\,dx = \frac{\pi^{1/2}2^{-v}y^{v+1/2}}{\Gamma(v+1/2)} \int_0^\infty \Big[\int_0^\infty (x^2-t^2)^{v-1/2}t^{1/2-v}f(t)\,dt\Big]\,e^{-xy}\,dx \tag{109}$$

for $Re(v) > -1/2$

$$\int_0^\infty f(x)K_v(xy)\,(xy)^{1/2}\,dx = \frac{\pi^{1/2}2^{-v}y^{1/2-v}}{\Gamma(v+1/2)} \int_0^\infty (t^2-y^2)^{v-1/2}\,\Big[\int_0^\infty x^{v+1/2}f(x)\,e^{-tx}\,dx\Big]\,dt \tag{110}$$

for $Re(v) > -1/2$

We selected to follow the path of lemma 4 and not the paths of relationships

(109) and (110), because lemma 4 provides us with more powerful and efficient schemes for our design. The same arguments apply for relation (111) that exists between Hankel and Laplace transforms as well as other existing relationships of the above transforms.

$$\int_0^\infty t^{v/2-1/4}\left[\int_0^\infty f(x)\,J_v(xy)\,(xy)^{1/2}\,dx\right]e^{-ts}\,dt \qquad (111)$$
$$= s^{-v-1}\int_0^\infty t^{v/2-1/4} f[(2t)^{1/2}]\,e^{-t/s}\,dt$$

The interconnections indicated in the above lemmas, play an important role in our design at stage two, thus avoiding redundancy and keeping the necessary knowledge down to a minimum.

Of course, the Integral transforms are not exhausted by the definitions that have been given so far. We have the Mellin transforms, the H transforms, the Kontorovich-Lebedev transforms and miscellaneous transforms such as the Fractional integrals, Hilbert etc. In general, the interconnection existing among them is loose and consequently each one requires individual attention. However, we should still look at each one in conjunction with the others. We will not make any more definitions nor we will extensively study here each particular transform - for more information on the different integral transforms see Sneddon and Bateman [18], [11].

We will next describe the ideas which are useful for our design (fig. 1), and applicable to each of the above mentioned transforms, preferably selecting examples with Laplace and Hankel transforms.

A design for the Integral Transform algorithm should incorporate two major components: the integration process, and the different Integral Transforms properties.

We decided to form a table which contains as few formulas as possible. This strategy has the following consequences:

1. The overall design of the system becomes algorithmic in the sense that the system works deterministically, knowing what it can really do and what it cannot, and does not waste time by trying different approaches.

2. The main burden and difficulty of the problem shifts from stage two to stages one and especially three, where we have to reduce the Generalized Hypergeometric Functions to some Elementary and/or Special Function(s).

As far as the Integral Transforms properties are concerned, our general policy consists of applying them in stage two, at the Generalized Hypergeometric Function level. Hence, stage two can be divided into two substages.

Substage 2.1 Utilize the Integral Transforms properties.

Substage 2.2 Integrate.

Let us first consider substage two and the decisions that must be made there.

The major decisions in substage two concern the contents of the table look-up. We decided that we should accept formulas in our table look-up only if they contain Generalized Hypergeometric Functions and then from this smaller set select the most general. Furthermore, lemmas one through six play a key role in deciding about the generality of such a formula.

However, we allow exceptions and we incorporate formulas that do not contain Generalized Hypergeometric Functions, under the following circumstances.

1. The Special Function(s) was (were) not successfully transformed into a Generalized Hypergeometric Function.

2. The expression resulting from stage one was not integrated at stage two.

For example, the Bessel function of the second kind $Y_n(z)$ for integer values of the index n, as we have already seen, is not expressible in terms of the Bessel function $J_v(z)$ and thus not in terms of a Generalized Hypergeometric Function. Inevitably, the cases where the function $Y_n(z)$, $n \in Z$, are involved require special consideration and their own formulas of integration. The same arguments apply for the Modified Bessel function of the second kind $K_n(z)$. Apart from $K_n(z)$, $Y_n(z)$, no Bessel function requires any particular attention, since every Bessel function is expressible in terms of $J_v(z)$, $Y_n(z)$ and $K_n(z)$, $v \in C$, $n \in Z$. A similar situation also exists for products of Special Functions that are not representable as Generalized Hypergeometric Functions.

Inability to integrate an expression at stage two is mainly caused by the fact that the argument, the number of parameters and the functional factor of the Generalized Hypergeometric Function do not meet the proper requirements and restrictions that the integration formula imposes.

For example, to apply formula (10) of Chapter 1, to an expression, the conditions

$$Re(s) > 0, \quad m+k < n+1, \text{ for integer values of } m, k \text{ and } n,$$

should be satisfied.

The "intermediate steps" policy we selected to follow in stage one is obviously of great help here and it obviously has support of very practical value. For example, whenever we utilize a function for our table look-up involving the Whittaker function $M_{\kappa,\mu}(z)$ and not a ${}_pF_q(z)$ we can automatically handle all special cases of $M_{\kappa,\mu}(z)$ too.

The key point in the formulas that do not involve Generalized Hypergeometric Functions - the "exception formulas" - is, to provide the most general representation in the table look up. Failure to do that will result in not finding integrable cases. This is a situation that occurs when the input expression contains a Special Function which is reducible to some other of a lower level. Since, we do not want to involve any reduction procedures in the first two stages, it becomes necessary to represent the "exception formulas" in their most general representation (even if mathematically they are equivalent).

General formulas at the Generalized Hypergeometric level are also incorporated in the design, for cases with finite intervals of integration [11].

Notice, however, that stage three is absolutely necessary and is utilized in every single one of the above mentioned cases. The incorporated formulas mentioned in the exception cases are still very general and consequently their outputs are even more so. Figure two below depicts the new version of figure one, describing the scheme in a more precise fashion.

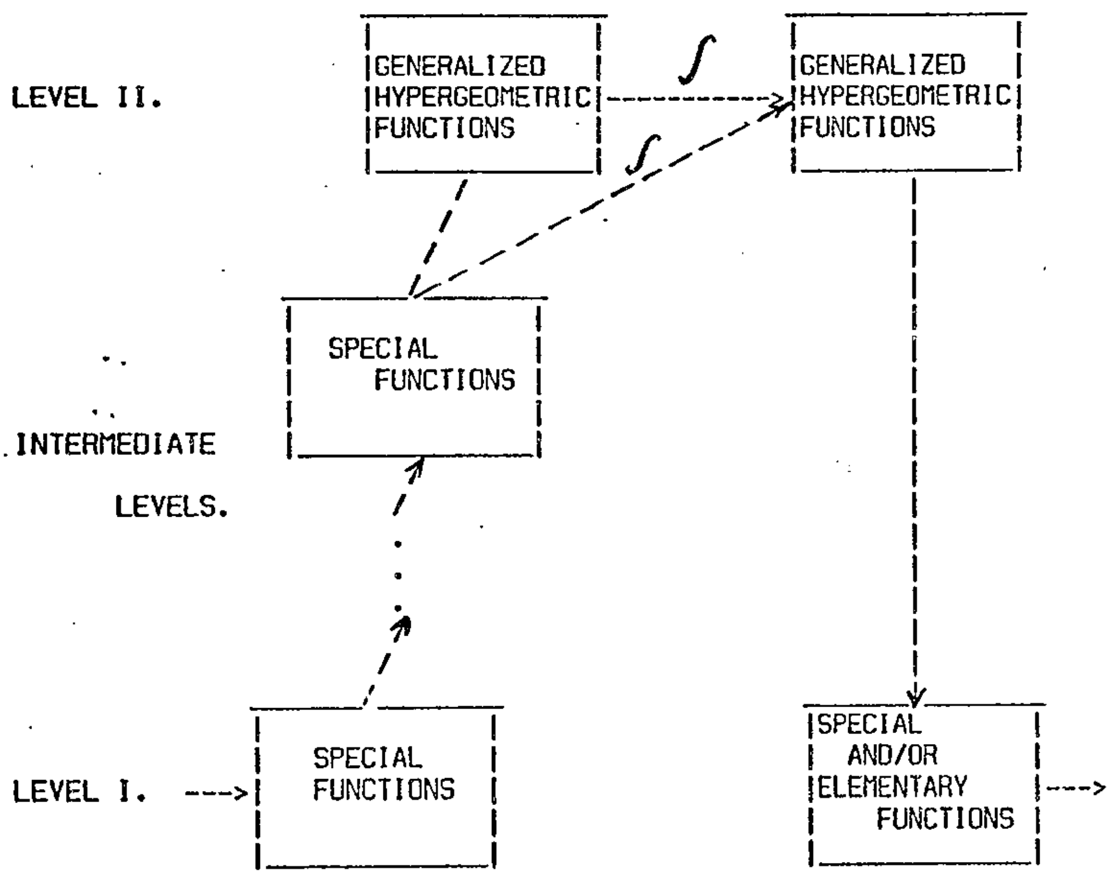

Figure 2.

Finally, we will consider the role of the different integral transform properties.

As we mentioned earlier, our general policy requires that any property should be applicable at the Generalized Hypergeometric Function level. In order to be more precise and incorporate the exceptions of the last paragraph, the different integral transform properties should be generally applied in the last level of generalization of the Special Functions, just before we are to look up the table and integrate. This policy changes only in cases where such a postponment of the application of the Integral transform property until stage two, causes irreparable damages in our procedure at stage two. Therefore, the Integral transforms properties have been considered in two types.

1. Properties that can be applied in substage 2.1, independently of what kind of Special Function(s) the input expression contains.

2. Properties that have to be applied after stage one for certain Special Functions.

Thus, for example, all the well known properties such as the "scale property" applicable to almost any integral transform is a type one property. Properties of the second type cannot be applied after stage one for certain Special Functions and our scheme is unable to proceed successfully to stages two and three. For example, the property

$$L[f(a\sinh t)] = \int_0^{\infty} J_p(au)g(u)\,du \qquad (112)$$

where $g(p) = L[f(t)]$, cannot be applied after stage one, for the Bessel function J_0, as in, for example,

$$J_0(a\sinh t)\,e^{-pt} \qquad (113)$$

since after the completion of the first stage we get

$$ {}_0F_1[\,1;\, -\frac{a^2}{4}\sinh^2 t]e^{-pt} \qquad (114)$$

Expression (114) cannot be integrated since our table does not contain any formulas with such functional arguments while it is too late to apply property (112).

This example could be solved by two recursive calls to our scheme (fig. 1). First, by calling the scheme as described for the Laplace Transforms, and second by calling the same scheme in which the Laplace Transforms properties and Integration formulas have been substituted with Hankel Transforms properties and Integration formulas [18]. Namely, our scheme will work as follows:

Given the integral

$$\int_0^\infty J_0(a\sinh t)e^{-pt}\,dt \qquad (115)$$

We first apply property (112) and we get

$$\int_0^\infty J_p(au)g(u)\,du \qquad (116)$$

where

$$g(u) = \int_0^\infty J_0(t)e^{-ut}\,du \qquad (117)$$

The first (Laplace transforms) call to our scheme (fig. 1), for the expression (117), gives

$$g(u) = (u^2+1)^{-1/2} \qquad (118)$$

The second (Hankel transforms) call to our scheme for the expression (116) given the result (118), provides

$$16^{v/2}a^{-1/2}\frac{\Gamma((v+1)/2)\Gamma(-v)(\Gamma(v/2+1))^2}{\Gamma(v+1)\Gamma((1-v)/2)}y^{1/2}J_{v/2}\left(\frac{ay}{2}\right)^2 \tag{119}$$

$$+\ a^{-1/2}\frac{\Gamma(v)\Gamma(v/2+1)\Gamma(1-v/2)}{\Gamma(v+1)}y^{1/2}J_{v/2}\left(\frac{ay}{2}\right)J_{-v/2}\left(\frac{ay}{2}\right)$$

Similarly, the Laplace transforms property

$$\int_0^\infty t^{v-1}f(t^{-1})e^{-pt}\,dt = p^{-v/2}\int_0^\infty u^{v/2}J_v(2u^{1/2}p^{1/2})g(u)\,du \tag{120}$$

where

$$g(u) = \int_0^\infty f(t)e^{-ut}\,dt \tag{121}$$

is a type two property.

Thus, the integral

$$\int_0^\infty t^{-1}J_1(t^{-1})e^{-pt}\,dt \tag{122}$$

is solved again by two calls to our scheme.

The first call, a Laplace transforms call, gives

$$g(u) = \int_0^\infty tJ_1(t)e^{-ut}\,dt = (u^2+1)^{-3/2} \tag{123}$$

and the second call, a Hankel transform call, provides the final result

$$\frac{\pi^{1/2}}{2\Gamma(3/2)}y^{3/2}J_1(y/4)K_1(y/4) \tag{124}$$

The Laplace property (125)

$$\int_0^\infty t^v f(t^2)e^{-pt}\,dt = 2^{-1/2}\pi^{-1/2}\int_0^\infty u^{v-2}e^{-1/4\,p^2u^2}D_v(pu)g(1/2\,u^{-2})\,du \tag{125}$$

where

$$g(p) = \int_0^\infty f(t)e^{-pt}\,dt \qquad (126)$$

is also a type two property.

On a first examination, a program that can take the Integral Transforms of approximately fifty Special Functions would imply it would be necessary that quite a large number of formulas be incorporated in the table look-up of our second stage. It turns out that relatively very few formulas are needed. For example, formula (10) of Chapter 1, is applicable to a large number of Special Functions [16], [3], [19], namely the Bessel functions of the first and second kind, both Modified Bessel functions, the two kinds of Hankel functions, also the Struve functions, the Lommel functions, and the Kelvin functions, the Whittaker, the error and both Incomplete Gamma functions, as well as to certain products of Bessel functions, for almost all the values of their indices and for linear as well as square roots of linear functions of the argument. Furthermore, in cooperation with general formulas of other Integral Transforms, formula (10) of Chapter 1, contributes in integrating composite functions like $J_0(\sinh t)$ and $t^{-1}J_1(t^{-1})$, as we have already shown.

Our main source of integration formulas has been the Bateman Manuscript Project which approximately contains 6000 integral formulas for the different transforms. Table 6 contains some key formulas for the Laplace transforms. Table 7 provides some key formulas applicable to both Laplace and K transforms.

$$\int_0^\infty t^{s-1}\,{}_mF_n(a_1,\dots,a_m;r_1,\dots,r_n;(lt)^k)e^{-pt}\,dt \tag{127}$$

$$= \Gamma(s)p^{-s}\,{}_{m+k}F_n\left(a_1,\dots,a_m,\frac{s}{k},\frac{s+1}{k},\dots,\frac{s+k-1}{k};r_1,r_2,\dots,r_n;\left(\frac{kl}{p}\right)^k\right)$$

$Re(s) > 0$, $m+k < n+1$, where k,m,n are integers.

$$\int_0^\infty x^{\mu-1}Y_v(xy)e^{-ax}\,dx \tag{128}$$

$$= -\frac{a^{v-\mu}\Gamma(\mu+v)\Gamma(\mu-v)}{2^{\mu-1}e^{v\pi i}\pi^{1/2}i^v\Gamma(\mu+1/2)}\,y^{-v}$$

$${}_2F_1\left[\frac{\mu-v}{2},\frac{\mu-v+1}{2};m+1/2;\frac{y^2+a^2}{a^2}\right]$$

$Re(\mu) > |Re(v)|$, $a > 0$

$$\int_0^\infty t^{M+v-1}e^{-At}\;{}_1F_1[\mu_1-\kappa_1;2\mu_1;a_1t] \tag{129}$$

$$\dots\;\dots\;\dots\qquad\dots\;\dots\;\dots$$

$${}_1F_1[\mu_n-\kappa_n;2\mu_n;a_nt]e^{-pt}\,dt$$

$$= (p+A)^{-v-M}\Gamma(v+M)$$

$$F_A[v+M;\mu_1-\kappa_1,\dots,\mu_n-\kappa_n;2\mu_1,\dots,2\mu_n;$$

$$a_1(p+A)^{-1},\dots,a_n(p+A)^{-1}]$$

$M = \mu_1+\dots+\mu_n$, $Re(v+M) > 0$, $A = 1/2(a_1+\dots+a_n)$

Table 6.

$$\int_0^\infty x^{\mu-3/2}\,{}_pF_q[a_1,\dots,a_p;b_1,\dots,b_q;-\lambda x^2]K_v(xy)(xy)^{1/2}\,dx \tag{130}$$

$$= 2^{\mu-2}y^{1/2-\mu}\Gamma((\mu+v)/2)\Gamma((\mu-v)/2)$$

$${}_{p+2}F_q[a_1,\dots,a_p,(\mu+v)/2,(\mu-v)/2;\beta_1,\dots,\beta_q;4\lambda y^{-2}]$$

$p \le q-1$, $Re(\mu) \ge |Re\ v|$

$$\int_0^\infty x^{v+1/2} {}_2F_1[\,\alpha,\ \beta;\ v+1;\ -\lambda^2x^2] K_v(xy)(xy)^{1/2}\ dx \quad (131)$$

$$= 2^{v+1}\lambda^{\alpha-\beta}y^{\beta-v-3/2}\Gamma(v+1)S_{1-\alpha-\beta,\alpha-\beta}(y/\lambda)$$

$\mathrm{Re}(\lambda) \geq 0, \quad \mathrm{Re}(v) \geq -1$

$$\int_0^\infty x^{4\mu+v-1/2}e^{-ax^2} {}_1F_1[\ 1/2+\mu-\kappa;\ 2\mu+1;\ ax^2] K_v(xy)\ dx \quad (132)$$

$$= 2^{\mu-\kappa-1/2}a^{-1/4-(3\mu+v+\kappa)/2}y^{\kappa-\mu-1}$$

$$\Gamma(2\mu+1)\Gamma(2\mu+v+1)e^{y^2/(8a)}W_{k,m}(y^2/(4a))$$

$2k = -3\mu-v-\kappa-1/2, \quad 2m = \mu+v-\kappa+1/2$

$\mathrm{Re}(\mu) \geq -1/2, \quad \mathrm{Re}(2\mu+v) > -1$

Table 7.

Hence, approximately thirty formulas are sufficient in conjuction with the other mathematical machinery we have mobilized, to exhaust all entries (around 500) in both Laplace and K transforms of the Bateman Manuscript. Actually, only a few of these formulas are needed to cover approximately 80% of the corresponding entries. Furthermore, numerous other cases can be solved that are not given explicitly in Bateman's tables. For example, expression (113) of this chapter does not match any entry of the Bateman's Tables. However, we succeeded in solving it by calling twice two different general formulas of the Bateman Manuscript.

However, the potentiality of keeping very few and general formulas around in the table of our second stage would be of limited value if we were unable to complete successfully the third stage, the reduction of the Generalized Hypergeometric Function to some Elementary and/or Special Function(s).

Chapter 3

THE REDUCTION STAGE

In the reduction stage the Generalized Hypergeometric Function is reduced, if that is possible, to some Elementary and/or Special Function(s). Priority is always given to those methods that reduce the Hypergeometric Series into elementary functions and then to those that reduce to the most common Special Functions, such as Error, Bessel, etc . The effort in the reduction stage increases as the number of the series parameters, and subsequently the p and q values, increase. If the reduction is unsuccessful then the series ${}_pF_q(z)$ is returned. Therefore, a complete reduction package should also incorporate schemes for the summation of the Generalized Hypergeometric Functions. In our thesis we did not extend our research in the "summation" domain. Of course, the reduction process will try to provide reducible forms of the Generalized Hypergeometric Series even if the series has a numeric value argument and it has been guaranteed that the series does not converge.

Example the Generalized Hypergeometric Function

$$ {}_2F_1[\, a,\ a + 1/2;\ c;\ -5] \qquad (1)$$

does not converge according to our convergence rules (see Chapter 2). The reduction process will nevertheless return:

$$\frac{2^{1-c}\, P_{c-2a-1,\,1-c}\left(\frac{1}{\mathrm{sqrt}(6)}\right)}{(-5)^{2a}\, 24^{\frac{1-c}{2}}\, \mathrm{gamma}(c)} \qquad (2)$$

where $P_{v,r}(z)$ is the Legendre function.

Similarly, if the argument is symbolic the reduction process will return an answer assuming proper intervals of validity for the symbolic references.

We feel that techniques similar to those in our reduction scheme can be also utilized in order to determine the closed form of a Generalized Hypergeometric Function with numerical argument. For a very interesting approach to the problem of series summation consult "A Calculus of Series Rearrangements" by Gosper [20].

Our approach to Definite Integration undoubtly also shows the strong interdependence that the problem of Definite Integration has with the problem of series summation.

In the "reduction" sections we provide our main conclusions and the necessary theory in terms of theorems, lemmas and corollaries. When necessary, algorithms and illustrations are provided.

3.1. GENERAL REDUCTION TESTS

In the general reduction part we perform reductions independent of the number of parameters.

Given the Generalized Hypergeometric Function ${}_pF_q[\,L_1;\ L_2;\ arg]$ where $L_1 = \{a_1, a_2, \ldots, a_p\}$, and $L_2 = \{b_1, b_2, \ldots, b_q\}$, the following lemmas hold:

Lemma 1. If $L = L_1 \cap L_2$ and $r = |L|$, then

$$ {}_pF_q[\,L_1;\ L_2;\ arg] = {}_{p-r}F_{q-r}[\,L_1-L;\ L_2-L;\ arg] \qquad (3)$$

The above lemma simply states that common numerator and denominator parameters can be eliminated with a subsequent subtraction in the subscripts p and q.

Lemma 2. If $-n \in L_1$, $n \in Z^+$, then

$$ {}_pF_q[\,L_1;\ L_2;\ arg] = 1 - \binom{n}{1} \frac{a_1a_2\ldots a_p}{b_1b_2\ldots b_q}\, arg + \qquad (4)$$

$$ \binom{n}{2} \frac{a_1(a_1+1)a_2(a_2+1)\ldots a_p(a_p+1)}{b_1(b_1+1)b_2(b_2+1)\ldots b_q(a_q+1)}\, arg^2 + \ldots $$

$$ (-1)^n \binom{n}{n} \frac{a_1(a_1+1)(a_1+2)\ldots(a_1+n-1)\ldots\ \ldots a_p(a_p+1)(a_p+2)\ldots(a_p+n-1)}{b_1(b_1+1)(b_1+2)\ldots(b_1+n-1)\ldots\ \ldots b_q(b_q+1)(b_q+2)\ldots(b_q+n-1)}\, arg^n $$

Hence, the Generalized Hypergeometric Function reduces to a polynomial.

Proof It can be easily seen that all terms after the nth one in the hypergeometric series expansion are zero. Hence, the series terminates and a polynomial is derived.

Corollary 2.1. If n in lemma 2 is zero the Generalized Hypergeometric Function reduces to one.

Lemma 3. If $n \in L_2$, $n \in Z^- \cup \{0\}$ then the Generalized Hypergeometric Function does not have any meaning.

Proof The denominators of the series expansion will contain zeroes after the nth term.

Of course lemma 3 always holds as long as no cancellation of the parameter n occurs. Hence, lemma 1 should always be applied first. This is not always true for lemma 2 over lemma 3.

Lemma 4. The polynomial expression (4) is an instance of some Orthogonal polynomial(s) according to the following cases:

a. If $|L_2| = 2$ and $|L_1| = 1$ then expression (4) reduces to one of the following polynomials:

i) Tchebichef, ii) Legendre, iii) Gegenbauer, iv) Jacobi.

b. If $|L_1| = |L_2| = 1$ then expression (4) reduces to one of the following polynomials:

i) Hermite, ii) Laguerre.

c. If $|L_1| = 2$ and $|L_2| = 0$ then expression (4) is also reducible to one of the polynomials mentioned in case b.

The three cases mentioned in lemma 4 will be investigated and discussed in the section concerned with special reduction tests, where our special reduction techniques will be used.

Of course, someone might suggest that lemma 4 and its anticipated discussions are redundant because lemma 2 itself is sufficient to successfully reduce the Generalized Hypergeometric Function. This is actually not true for the following reasons:

1. If the polynomial $p_2(x)$ is returned instead of its expansion, the reader becomes instantly aware that he is now dealing with a particular polynomial, the Orthogonal polynomial of Legendre, and not an arbitrary one.

2. If, say, the Tchebyshev polynomial $T_n(x)$, $n \in N$, is returned instead of its expansion we are certain that no "blow up" is likely to occur as a result of exceeding the storage capacity of a machine. $T_n(x)$ will be returned very quickly. Its expansion for a large n, $n \in N$, will either exceed storage capacity or a huge polynomial will be returned which is difficult to comprehend and further manipulate (i.e. differentiate, factor etc.).

For those cases in which it is not possible that an Orthogonal polynomial be returned and in which n is relatively large, our implementation will first return a warning and then return the Generalized Hypergeometric Function instead. It will take chances and try to return the expansion only if ordered to do so. However, such a situation is very unlikely to occur in any "real" problem.

Lemma 5. If a numerator parameter of the Generalized Hypergeometric Series ${}_pF_q(z)$ exceeds a denominator parameter by a positive integer, say k, then the series ${}_pF_q(z)$ can be expressed as the sum of k+1 ${}_{p-1}F_{q-1}(z)$'s. Specifically, the following relation holds:

$$
\begin{aligned}
&{}_pF_q[\, a_1+k,\ a_2,\ldots,\ a_p;\ a_1,\ b_2,\ b_3,\ldots,\ b_q;\ z] = \qquad (5)\\
&\binom{k}{0}\ {}_{p-1}F_{q-1}[\, a_2,\ a_3,\ldots,\ a_p;\ b_2,\ b_3,\ldots,\ b_q;\ z] +\\
&\binom{k}{1}\ \frac{a_2a_3\ldots a_p}{a_1b_2b_3\ldots b_q}\\
&\qquad z\ {}_{p-1}F_{q-1}[\, a_2+1,\ a_3+1,\ldots,\ a_p+1;\ b_2+1,\ b_3+1,\ldots,b_q+1;\ z] +\\
&\binom{k}{2}\ \frac{a_2(a_2+1)a_3(a_3+1)\ldots a_p(a_p+1)}{a_1(a_1+1)b_2(b_2+1)\ldots b_q(b_q+1)}\\
&\qquad z^2\ {}_{p-1}F_{q-1}[a_2+2,a_3+2,\ldots,a_p+2;b_2+2,b_3+2,\ldots,b_q+2;\ z] +\\
&\ldots \quad \ldots \quad \ldots\\
&\binom{k}{k}\ \frac{a_2(a_2+1)\ldots(a_2+k-2)(a_2+k-1)\ldots\ \ldots a_p(a_p+1)\ldots(a_p+k-2)(a_p+k-1)}{a_1(a_1+1)\ldots(a_1+k-2)(a_1+k-1)\ldots\ \ldots b_q(b_q+1)\ldots(b_q+k-2)(b_q+k-1)}\ z^k\\
&\qquad {}_{p-1}F_{q-1}(\, a_2+k,\ a_3+k,\ \ldots\ ,\ a_p+k;\ b_2+k,\ b_3+k,\ \ldots\ ,\ b_q+k;\ z)
\end{aligned}
$$

Proof By induction.

Lemma 5 constitutes a surprisingly useful rule, which is incorporated in the first reduction phase. Such a series splitting, though it does not actually fully reduce a ${}_pF_q(z)$, simplifies the reduction by decreasing the p and q values. We illustrate our ideas in the example 1.

Example 1. Consider

$$t^3\ J_0(t^{1/2})^2\ e^{-pt} \qquad (6)$$

after stages one and two have been completed, we get

$$6p^{-4}\ {}_3F_3[\ 1/2,\ 1,\ 4;\ 1,\ 1,\ 1;\ p^{-1}] \qquad (7)$$

now, at stage three and after a trivial general reduction rule, expression (7) becomes

$$6p^{-4}\ {}_2F_2[\ 1/2,\ 4;\ 1,\ 1;\ -p^{-1}] \qquad (8)$$

then applying our general "splitting" rule, (8) reduces to

$$6p^{-4}\ [\ {}_1F_1[\ 1/2;\ 1;\ -p^{-1}]\ -\ 3/2\ p^{-1}\ {}_1F_1[\ 3/2;\ 2;\ -p^{-1}]\ + \qquad (9)$$
$$+\ 9/16\ p^{-2}\ {}_1F_1[\ 5/2;\ 3;\ -p^{-1}]\ -\ 5/96\ p^{-3}\ {}_1F_1[\ 7/2;\ 4;\ -p^{-1}]]$$

which ultimately yields:

$$6p^{-4}\ e^{-1/2\ p^{-1}}\ [\ I_0(-1/2\ p^{-1})\ +\ 3/2\ M_{-1/2,1/2}(-p^{-1})\ + \qquad (10)$$
$$+\ 9/16\ p^{-2}\ (-p)^{3/2}\ M_{-1,1}(-p^{-1})\ -\ 5/96\ p^{-1}\ M_{-3/2,3/2}(-p^{-1})]$$

where $M_{i,j}$ is a Whittaker function.

We proceed with the algorithm GR for the general reduction part.

Algorithm GR. Given

$$ {}_pF_q[\ L_1;\ L_2;\ z] \qquad (11)$$

Step 1. Find the intersection L, of the two parameter lists L_1 and L_2. If it is nonempty substitute (11) with

$$ {}_{p-|L|}F_{q-|L|}[\ L_1-L;\ L_2-L;\ z] \qquad (12)$$

Step 2. If any numerator parameter, exceeds a denominator parameter by a positive integer k, then return k+1 ${}_{p-1}F_{q-1}(z)$'s to be processed by Algorithm GR.

Step 3. Search for a nonpositive integer n in L_1 list.

Step 4. Search for a nonpositive integer n' in L_2 list.

Step 5. If n, n' found then

If $n \geq n'$ then return the

Generalized Hypergeometric Function.

else

If n' only found then return the

Generalized Hypergeometric Function.

Step 6. If $|L_1| = 2$ and $|L_2| = 1$

or $|L_1| = 1 = |L_2|$

or $|L_1| = 2$ and $|L_2| = 0$

then dispatch to the special reduction tests algorithms,

else return a polynomial of degree n according to the formula (4).

Step 7. Return the Generalized Hypergeometric Function.

3.2. SPECIAL REDUCTION TESTS

The Special Reduction tests constitute the second phase of the reduction stage. Here, different algorithms are performed which are dependent on the values of p and q of the Generalized Hypergeometric Function. The general idea in this phase is to divide the set of Special and Elementary Functions into subsets depending on the p and q values that their series representation has. Therefore, algorithms are constructed according to the particular subset of Special and Elementary Functions and the available mathematical machinery applicable to the subset. It should be noted that these algorithms search first to return Elementary or common Special Functions. In case that they completely fail, the series is returned. The most important tools here are differentiation, the different transformations such as linear, quadratic etc. and the contiguous functions relations. Differentiation and contiguous functions relations can be utilized in any subset independently of the values of p and q. However, this is not true for the different transformations (linear, quadratic, etc.). A characteristic of the differential and contiguous relations is their ability to transform the Generalized Hypergeometric Functions into some new ones which are associates[1] of the old. Hence, differential or contiguous relations are utilized whenever parameters in the Generalized Hypergeometric Function differ with the corresponding series of our table by some integer quantity. Similarly, transformations (quadratic, cubic etc.) are applicable to a Generalized

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[1] The series ${}_pF_q[\, a_1+m_1, \ldots, a_p+m_p;\ b_1+n_1, \ldots, b_q+n_q;\ z]$ for m_i, n_j $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$ integer numbers, are called the <u>associates</u> series.

Hypergeometric Function whenever some particular relationship holds among its parameters.

In the second phase, reductions are easy for the cases ${}_0F_0(z)$, ${}_0F_1(z)$, ${}_1F_0(z)$, and the difficulty increases significantly for higher p's and q's. We have been mainly concerned with the Confluent Hypergeometric Functions reduction, ${}_1F_1(z)$, and the Gauss Hypergeometric Functions, ${}_2F_1(z)$, that include in addition to certain important Special Functions, the Elementary Functions.

3.2.1. EXPONENTIAL BINOMIAL AND BESSEL REDUCTIONS

Lemma 6. For $z \in C$ relation (12) holds

$$ {}_0F_0[\ ;\ ;\ z] = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots = e^z \quad (12) $$

Hence, in case of ${}_0F_0(z)$ as lemma 6 indicates, we have all the exponential, trigonometric and hyperbolic functions, since all of their series expansion is of type (12).

Lemma 7. For any a, $z \in C$ relation (13) holds

$$ {}_1F_0[\ a;\ ;\ z] = 1 + az + \frac{a(a+1)}{2!} z^2 + \dots + \frac{(a)_n}{n!} z^n + \dots = (1-z)^{-a} \quad (13) $$

thus category ${}_1F_0(z)$ includes the Binomial functions.

Lemma 8. For any $v,\ z \in C$ relations (14), (15) hold

$$ {}_0F_1[\ ; v+1;\ -z^2/4] = (z/2)^{-v}\ \Gamma(v+1)\ J_v(z) \qquad (14)$$

$$ {}_0F_1[\ ; v+1;\ z^2/4] = (z/2)^{-v}\ \Gamma(v+1)\ I_v(z) \qquad (15)$$

particularly for v equal to 1/2, -1/2, 3/2, -3/2 relation (14) - and similarly (15) - reduces to the following relations

$$ {}_0F_1[\ ;\ 3/2;\ z^2/4] = 2\ \Gamma(3/2)\ \pi^{-1/2}\ z^{-1}\ \sin(z) \qquad (16)$$

$$ {}_0F_1[\ ;\ 5/2;\ z^2/4] = 4\ \Gamma(5/2)\ \pi^{-1/2}\ (z^{-1}\ \sin(z) - \cos(z)) \qquad (17)$$

$$ {}_0F_1[\ ;\ 1/2;\ z^2/4] = \Gamma(1/2)\ \pi^{-1/2}\ \cos(z) \qquad (18)$$

$$ {}_0F_1[\ ;\ -1/2;\ z^2/4] = -1/2\Gamma(-1/2)\pi^{-1/2})z(z^{-1}\cos(z) + \sin(z)) \qquad (19)$$

In particular, the following important theorem holds

Theorem 1.[1] The Bessel function $J_v(z)$, is expressible in finite terms by means of algebraic and trigonometric funcions of z, when v is half of an odd integer. Namely, the following relations are true

$$ J_{n+1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \Bigg[\sin(z-n\pi/2) \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r (n+2r)!}{(2r)!(n-2r)!(2z)^{2r}} + \cos(z-n\pi/2) \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^r (n+2r+1)!}{(2r+1)!(n-2r-1)!(2z)^{2r+1}}\Bigg] \qquad (20)$$

$$ J_{-n-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \Bigg[\cos(z+n\pi/2) \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r (n+2r)!}{(2r)!(n-2r)!(2z)^{2r}} \qquad (21)$$

---

[1] Based on the work of Lommel, Studien uber die Bessel'schen Funktionen(Leipzig, 1968), pp. 51-56.

$$- \sin(z+n\pi/2) \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^r (n+2r+1)!}{(2r+1)!(n-2r-1)!(2z)^{2r+1}}]$$

However, when v does not have such a value, then the Bessel function $J_v(z)$ is not so expressible. This fact has been established by Liouville's theorem which we describe next.

<u>Theorem 2.</u>[1] The Bessel's equation for functions of order v has no nontrivial solution expressible in finite terms by means of elementary transcendental functions, if $2v$ is not an odd integer.

3.2.2. CONFLUENT HYPERGEOMETRIC FUNCTION REDUCTIONS

<u>Lemma 9.</u> The following relation holds

$$ {}_1F_1[\, a;\ c;\ z] = e^z\ {}_1F_1[\, c-a;\ c;\ -z] \qquad (22)$$

The above linear transformation is also known as a Kummer's transformation.

<u>Lemma 10.</u> Given ${}_1F_1[\, a;\ c;\ z]$ such that a-c is an integer number then ${}_1F_1[\, a;\ c;\ z]$ is reducible to the exponential or the Binomial or the Error or the Incomplete Gamma function.

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[1] Journal de Math. VI (1841).

<u>Proof</u> We distinguish the following cases:

1. If a-c=0, then

$${}_1F_1[\, a;\ c;\ z] = {}_1F_1[\, a;\ a;\ z] = {}_0F_0[\ ;\ ;z] = e^z \qquad (23)$$

2. If a-c is a positive integer, then from lemma 9 and lemma 2 we get:

$${}_1F_1[\, a;\ c;\ z] = e^z\, {}_1F_1[\, c-a;\ c;\ -z] = e^z\ (1+z)^{a-c} \qquad (24)$$

since c-a is a negative integer.

3. If a-c is a negative integer, we distinguish the following cases:

3.1. If c = 1 then a must be 0. Hence, and according to the Corollary 2.1

$${}_1F_1[\, 0;\ 1;\ z] = 1 \qquad (25)$$

3.2. Let us assume now: a-c = -m, m = 1,2,3,... , then we proceed with the following inductive argument:

<u>Base</u>: If c-a=1, then

1. If a=1/2 and c=3/2 then because

$$\mathrm{erf}(x) = 2\pi^{-1/2}x\ {}_1F_1[\, 1/2;\ 3/2;\ -x^2] \qquad (26)$$

it follows that ${}_1F_1[\, a;\ c;\ z]$ reduces to the Error Function.

2. If a≠1/2 and/or c≠3/2 then because

$$\gamma(a,x) = a^{-1}x^a\ {}_1F_1[\, a;\ a+1;-x] \qquad (27)$$

holds true, it follows that ${}_1F_1[\, a;\ c;\ z]$ reduces to the Incomplete Gamma function.

Assume now that for: c-a = m, ${}_1F_1[\, a;\ c;\ z] = {}_1F_1[\, a;\ a+m;\ z]$ reduces to the Error or the Incomplete Gamma function. We will prove that this is also true for: c-a = m+1. (28)

Utilizing the contiguous function relation:

$$(a-c+1)\ {}_1F_1[\,a;\ c;\ z] - a\ {}_1F_1[\,a+1;\ c;\ z] + (c-1)\ {}_1F_1[\,a;\ c-1;\ z] = 0 \qquad (29)$$

and substituting (28) into (29) we get

$${}_1F_1[\,a;\ a+m+1;\ z] = -a\ m^{-1}\ {}_1F_1[a+1;a+m+1;\ z] + m^{-1}(a+m)\ {}_1F_1[\,a;a+m;\ z] \qquad (30)$$

thus the right hand side of expression (30) is expressed as a linear combination of two other Generalized Hypergeometric Functions which both have the property of having: $c-a = m$ and for which case our induction hypothesis is valid.

<u>Corollary 10.1.</u> For any complex a and nonnegative integer m the following relation is true

$${}_1F_1[\,a;\ a+m;\ z] = \qquad (31)$$

$$\binom{m-1}{0}\frac{(a+1)(a+2)\ldots(a+m-1)}{(m-1)!}\ {}_1F_1[\,a;\ a+1;\ z]\ -$$

$$\binom{m-1}{1}\frac{a(a+2)\ldots(a+m-1)}{(m-1)!}\ {}_1F_1[\,a+1;\ a+2;\ z]\ +$$

$$\binom{m-1}{2}\frac{a(a+1)(a+3)\ldots(a+m-1)}{(m-1)!}\ {}_1F_1[\,a+2;\ a+3;\ z]\ -$$

$$\ldots \quad \ldots \quad \ldots$$

$$(-1)^{m-1}\binom{m-1}{m-1}\frac{a(a+1)(a+2)\ldots(a+m-2)}{(m-1)!}\ {}_1F_1[\,a+(m-1);\ a+m;\ z]$$

<u>Proof</u> Relation (31) can be easily proved by induction on m, given relation (29).

By comparing relations (26) and (27) we can see that

$$\mathrm{erf}(x) = 1/2\ \gamma(1/2,x^2) \qquad (32)$$

Hence, the Error function is an instance of the Incomplete Gamma function. Therefore, our reduction procedure should be able to return the Error functions instead of the Incomplete Gamma function, whenever this is the case. Lemma 11 and lemma 12 along with lemma 10 complete the reduction for the Incomplete Gamma and the Error function cases.

<u>Lemma 11.</u> For m and n nonnegative integers the following relations hold:

1. For $m \geq n$,

$$ {}_1F_1[\,1/2+n;\ 3/2+m;\ x] = (-1)^{m-n} \frac{(3/2)_{m-n}\,(m-n+3/2)_n}{(1)_{m-n}(1/2)_n} \frac{d^n}{dx^n}\Big[e^x \frac{d^{m-n}}{dx^{m-n}}\big[e^{-x}\,{}_1F_1[\,1/2;\ 3/2;\ x]\big]\Big] \quad (33) $$

2.

$$ {}_1F_1[\,1/2-n;\ 3/2+m;\ x] = (-1)^m \frac{(3/2)_m\, e^x}{(1)_m\,(1+m)_n\, x^m} \frac{d^n}{dx^n}\Big[\,x^{m+n} \frac{d^m}{dx^m}\big[\,e^{-x}\,{}_1F_1[\,1/2;\ 3/2;\ x]\big]\Big] \quad (34) $$

3. For $n \geq m$,

$$ {}_1F_1[\,1/2-n;\ 3/2-m;\ x] = (-1)^m \frac{e^x\, x^m}{(1-m)_n(-1/2)_m} \frac{d^n}{dx^n}\Big[\,e^{-x}\,x^{n-1/2} \frac{d^m}{dx^m}\big[\,x^{1/2}\,{}_1F_1[\,1/2;\ 3/2;\ x]\big]\Big] \quad (35) $$

<u>Proof</u> Relation (33) can be easily proved from relations (34) and (37) of Chapter 2. Likewise, relation (34) can be deduced from relation (37) and (38) of Chapter 2, and finally, relation (35) follows from relations (36) and (38) of Chapter 2.

Lemma 12. Given the conditions of lemma 10 and furthermore assuming a and c are not symbolic quantities then ${}_1F_1[\ a;\ c;\ z]$ is reducible to the exponential or the Binomial or the Error functions.

Proof The proof is a consequence of lemmas 10 and 11.

Particular cases of the Confluent Hypergeometric Functions set are the Orthogonal Polynomials of Hermite and Laguerre as the following lemma describes.

Lemma 13. The Generalized Hypergeometric Function ${}_1F_1[\ -n;\ a;\ x]$ reduces for any $n \in Z^+$, $a \in C$ into Hermite or Laguerre polynomials.

Proof From relations (36), (37) and (38)

$$He_{2n}(x) = (-2)^n\ (1/2)_n\ {}_1F_1[\ -n;\ 1/2;\ x^2/2] \tag{36}$$

$$He_{2n+1}(x) = (-2)^n\ (3/2)_n \times {}_1F_1[\ -n;\ 3/2;\ x^2/2] \tag{37}$$

$$L_{n,a}(x) = \binom{n+a}{n}\ {}_1F_1[\ -n;\ a+1;\ x] \tag{38}$$

we can easily deduce the assertion.

If we compare (36) and (37) with (38) we can conclude that Hermite polynomials are particular cases of the Laguerre polynomials, particularly the following relations hold:

$$He_{2n}(x) = (-2)^n\ n!\ L_{n,-1/2}(x^2/2) \tag{39}$$

$$He_{2n+1}(x) = (-2)^n\ n! \times L_{1/2,n}(x^2/2) \tag{40}$$

The reader might have expected by now that a lemma similar to lemma 11, holding for the Error function, should have been presented here to increase the reduction capabilities to Hermite polynomials. Of course we could have proved similar to lemma 11 reduction relations for the Hermite polynomials but we would not have increased at all the capabilities of our reduction because differentiation of Hermite polynomials provides Laguerre ones. Hence, and after a little inspection of the expression (38) the redundancy of such a lemma here is obvious.

Bessel functions are also particular cases of Confluent Hypergeometric Functions as lemma 14 states.

Lemma 14. The following relation is true:

$$ {}_1F_1[\ a;\ 2a;\ 2z] = e^z\ {}_0F_1[\ ;\ a+1/2;\ z^2/4] \tag{41}$$

In other words, a Confluent Hypergeometric Function ${}_1F_1[\ a;\ c;\ z]$ reduces to a Bessel type of function if

$$ c = 2a \tag{42}$$

Theorem 3. Given an integer number a then the Confluent Hypergeometric Function ${}_1F_1[\ a;\ c;\ z]$ reduces to the exponential the Binomial or the Error or the Incomplete Gamma functions.

Proof Assume a = m, where m = 1,2,3,... then

$$ {}_1F_1[\ m;\ c;\ z] = {}_1F_1[\ m;\ (c-m)+m;\ z] \tag{43}$$

expression (43) and lemma 9 give:

$$ {}_1F_1[\ m;\ c;\ z] = e^z\ {}_1F_1[\ c-m;\ (c-m)+m;\ -z] = e^z\ {}_1F_1[a';c';-z] \tag{44}$$

where a'= c-m, c'= (c-m)+m and it is a'-c' = -m. Hence, lemmas 10 and 14 apply to the right hand side of relation (44) and the lemma is proved.

Lemma 16. Any Confluent Hypergeometric Function ${}_1F_1[\,a;\ c;\ z]$ can be expressed as a Whittaker function $M_{\kappa,\mu}(z)$.

Proof This is obvious from relation (45).

$$M_{\kappa,\mu}(z) = z^{\mu+1/2}\ e^{-z/2}\ {}_1F_1[\ 1/2+\mu-\kappa;\ 2\mu+1;\ z] \qquad (45)$$

Hence, any Special Function which is in particular a Confluent Hypergeometric Function is an instance of the Whittaker function $M_{\kappa,\mu}(z)$.

Algorithm "${}_1F_1$-Red" depicts our conclusions of the present subsection.

Algorithm ${}_1F_1$-Red. Given ${}_1F_1[\ a;\ c;\ z]$

Step 1. If $2a = c$ then return to ${}_0F_1$ cases

Step 2. If $a - c \notin Z$ then go to Step 7

Step 3. If $a - c = 0$ then go to gen-red algorithm

Step 4. If $a - c > 0$ then apply transformation (22) and go to Step 6

Step 5. If a and/or c not numeric quantities
then return the Incomplete Gamma function $\gamma(a,x)$
else return the Error function Erf(z)

Step 6. If c is 1/2 or 3/2 then return the Hermite polynomial
else return the Laguerre polynomial

Step 7. If $a \in Z$ then apply transformation (22) and go to Step 3
else return the Whittaker function $M_{\kappa,\mu}(z)$.

3.2.3. GAUSS HYPERGEOMETRIC FUNCTIONS REDUCTIONS

In this subsection special reductions are performed that lead to Elementary Functions as well as to the Special Functions of Legendre, Incomplete Beta and Orthogonal polynomials of Tchebichef, Legendre, Gegenbauer and Jacobi. The machinery that is utilized for reduction is the contiguous function relations, differentiation, and different transformations (linear, quadratic etc.)

3.2.3.1. THE LIMIT-REDUCTION METHOD

One of our methods in accomplishing reduction is the "limit-reduction" method. Our method works in the following fashion:

Algorithm I.

Given the hypergeometric function:

$$ {}_2F_1[\ a,\ b;\ c;\ z] \tag{46} $$

Step 1. Detect that the hypergeometric function (46) is the limit of some other hypergeometric function, as far as their parameter part is concerned, which other hypergeometric function can be processed by a quadratic transformation. Namely, a relation

$$\lim_{\substack{r_1 \to l_1 \\ \dots \\ r_n \to l_n}} {}_2F_1[\, a', b'; c'; z'] = {}_2F_1[\, a, b; c; z'] \tag{46A}$$

holds, where r_i $i = 1, 2, \dots, n$ are quantities involved in the parameters a', b' and c' and furthermore

$${}_2F_1[\, a', b'; c'; z'] = f(x)\,{}_2F_1[\, a'', b''; c''; z'] \tag{46B}$$

is the quadratic transformation, where z and z' are assumed arguments of the variable x. (Relation (46B) is not any particular quadratic transformation, it is just a general "picture" of any quadratic transformation which help us express the algorithm in a better way.)

<u>Step 2.</u> Take the same limits on this quadratic transformation. Thus take

$$\lim_{\substack{r_1 \to l_1 \\ \dots \\ r_n \to l_n}} {}_2F_1[\, a', b'; c'; z'] \tag{46C}$$

$$= \lim_{\substack{r_1 \to l_1 \\ \dots \\ r_n \to l_n}} f(x) \lim_{\substack{r_1 \to l_1 \\ \dots \\ r_n \to l_n}} {}_2F_1[\, a'', b''; c''; z']$$

$$=: g(x)\; {}_2F_1[\, a''', b'''; c'''; z']$$

<u>Step 3.</u> Call the "reduction algorithm" to process the hypergeometric function:

$${}_2F_1[\, a''', b'''; c'''; z'] \tag{46D}$$

and multiply the result with g(x).

<u>Step 4.</u> In case that $z \neq z'$ perform the appropriate adjustment in the result of step three and thus provide the reduced result of the hypergeometric function (46).

Of course, "limit-reduction" is applied whenever we are certain that the second call to the reduction algorithm at Step 4, will be successful. The above method is utilized to reduce the hypergeometric function into Binomial or Elementary Functions. We illustrate this method with a simple example.

We are given the hypergeometric function:

$$ {}_2F_1[\,3/4,\ 5/4;\ 1/2;\ z^2] \tag{47}$$

<u>Step 1.</u> It is true that:

$$ \lim_{\substack{a\to 3/2\\ b\to 0}} {}_2F_1[\,a/2,\ 1/2+a/2;\ b+1/2;\ z^2] = {}_2F_1[\,3/4,\ 5/4;\ 1/2;\ z^2] \tag{48}$$

therefore the following quadratic transformation is applicable

$$ {}_2F_1[\,a/2,\ 1/2+a/2;\ b+1/2;\ z^2/(2-z)^2] = (1 - z/2)^a\ {}_2F_1[\,a,\ b;\ 2b;\ z] \tag{49}$$

<u>Step 2.</u> We next take the limits of relation (49) and we get

$$ \begin{aligned} &\lim_{\substack{a\to 3/2\\ b\to 0}} {}_2F_1[\,a/2,\ 1/2+a/2;\ b+1/2;\ z^2/(2-z)^2] \\ &\quad = \lim_{\substack{a\to 3/2\\ b\to 0}} (1 - z/2)^a\ {}_2F_1[\,a,\ b;\ 2b;\ z] \\ &\qquad = (1-z/2)^{3/2}\ {}_2F_1[\,3/2,\ 0;\ 0;\ z] \end{aligned} \tag{50}$$

<u>Step 3.</u> Processing of the right hand side of the expression (50) by the Reduction algorithm we get

$$ \left(\frac{2-z}{2(1-z)}\right)^{3/2} \tag{51}$$

<u>Step 4.</u> And finally adjusting the arguments we have

$$\frac{1}{6(1+z)^{3/2}} + \frac{5}{6(1-z)^{3/2}} \qquad (52)$$

We next pay a closer look for the above proposed "limit-reduction" method.

Lemma 17. Given the hypergeometric function ${}_2F_1[\, a,\ b;\ c;\ z]$, then there exists a quadratic transformation if and only if the numbers

$$\pm(1-c),\ \pm(a-b),\ \pm(a+b-c) \qquad (53)$$

have the property that one of them equals 1/2 or that two of them are equal.

Lemma 17 constitutes the criterion for accepting a given hypergeometric function for further manipulation by several reduction methods involving quadratic transformations. Of course, the limit reduction method is one of them. As it finally turns out, there is always a way to reduce a hypergeometric function which meets the requirements of lemma 17.

A table of all the existing quadratic transformations is provided in the Appendix 2.

Lemma 18. The following relations hold:

$${}_2F_1\left[\, a/2,\ -a/2;\ 1/2;\ \frac{z^2}{4(z-1)}\right] = (1-z)^{-a/2} \qquad (54)$$

$${}_2F_1\left[\, \frac{1-a}{2},\ \frac{1+a}{2};\ 1/2;\ \frac{z^2}{4(z-1)}\right] = \frac{2(1-z)^{1-a/2}}{2-z} \qquad (55)$$

$${}_2F_1\left[a/2, \frac{a+1}{2}; 1/2; \frac{z^2}{(2-z)^2}\right] = \frac{(2-z)^a}{2^a(1-z)^a} \quad (56)$$

$${}_2F_1\left[c/2, \frac{c-1}{2}; c; 4z(1-z)\right] = (1-z)^{1-c} \quad (57)$$

$${}_2F_1\left[c/2, \frac{c+1}{2}; c; 4z(1-z)\right] = \frac{(1-z)^{1-c}}{1-2z} \quad (58)$$

$${}_2F_1[c-1, c-1/2; 2c-1; 4z^{1/2}(1+z)^{1/2}((1+z)^{1/2}+ z^{1/2})^{-2}] = \quad (59)$$
$$= ((1 + z)^{1/2} + z^{1/2})^{2c-2} (1+z)^{1-c}$$

$${}_2F_1\left[c, c-1/2; 2c-1; \frac{4z^{1/2}(1+z)^{1/2}}{[(1+z)^{1/2}+z^{1/2}]^2}\right] \quad (60)$$
$$= (1+z)^{1-c}(z^{1/2} + (1+z)^{1/2})^{2c}$$

<u>Proof</u> Utilizing the following quadratic transformations

$${}_2F_1[a, b; 2b; z] = (1-z/2)^{-a}\, {}_2F_1\left[a/2, 1/2+a/2; b+1/2; \frac{z^2}{(2-z)^2}\right] \quad (61)$$

$${}_2F_1[a, 1-a; c; z] = (1-z)^{c-1}{}_2F_1[c/2-a/2, (c+a-1)/2; c; 4z(1-z)] \quad (62)$$

$${}_2F_1[a, 1-a; c; -z] = (1+z)^{c-1} ((1+z)^{1/2} + z^{1/2})^{2-2a-2c} \quad (63)$$
$${}_2F_1\left[c+a-1, c-1/2; 2c-1; \frac{4z^{1/2}(1+z)^{1/2}}{[(1+z)^{1/2}+z^{1/2}]^2}\right]$$

we can deduce the following relations

$$\lim_{b\to 0} {}_2F_1\left[a/2, b-a/2; b+1/2; \frac{z^2}{4(z-1)}\right] = (1 - z)^{-a/2} \quad (64)$$

$$\lim_{b\to 0} {}_2F_1\left[\frac{1-a+2b}{2}, \frac{1+a}{2}; b+1/2; \frac{z^2}{4(z-1)} \right] = \frac{2(1-z)^{(1-a)/2}}{2-z} \tag{65}$$

$$\lim_{b\to 0} {}_2F_1\left[a/2, \frac{a+1}{2}; b+1/2; \left(\frac{z}{2-z}\right)^2 \right] = \frac{(2-z)^a}{2^a(1-z)^a} \tag{66}$$

$$\lim_{a\to 0} {}_2F_1\left[c/2-a/2, \frac{c+a-1}{2}; c; 4z(1-z) \right] = (1-z)^{1-c} \tag{67}$$

$$\lim_{a\to 0} {}_2F_1\left[c/2+a/2, \frac{c-a+1}{2}; c; 4z(1-z) \right] = \frac{(1-z)^{1-c}}{1-2z} \tag{68}$$

$$\lim_{a\to 0} {}_2F_1\left[c+a-1, c-1/2; 2c-1; \frac{4z^{1/2}(1+z)^{1/2}}{((1+z)^{1/2}+z^{1/2})^2} \right] = \tag{69}$$

$$= ((1+z)^{1/2} + z^{1/2})^{2c-2}(1+z)^{1-c}$$

$$\lim_{a\to 0} {}_2F_1\left[c-a, c-1/2; 2c-1; \frac{4z^{1/2}(1+z)^{1/2}}{((1+z)^{1/2}+z^{1/2})^2} \right] \tag{70}$$

$$= (1+z)^{1-c}(z^{1/2}+(1+z)^{1/2})^{2c}$$

Hence, relations (54) through (60) hold.

As it should be noticed, Lemma 18 is nothing more than the application of Algorithm I, into the elligible subset of quadratic transformations for limit reduction. Equivalent or special cases for some of the relations (59) through (60) are given in the Appendix 3, which has been selected from the existing literature [3], [21] and [22].

Lemma 18 can be further generalized by incorporating into it differential relations. Thus the hypergeometric function

$${}_2F_1[a, a+1/2; 3/2; z] \tag{71}$$

by means of the differential relation

$$\frac{(c-a)_n (c-b)_n}{(c)_n} (1 - z)^{a+b-c-n} \, {}_2F_1[\, a,\ b;\ c+n;\ z] \tag{72}$$

$$= \frac{d^n}{dz^n} [(1\ z)^{a+b-c} \, {}_2F_1[\, a,\ b;\ c;\ z]]$$

can be reduced to the hypergeometric function

$${}_2F_1[\, a,\ a+1/2;\ 1/2;\ z] \tag{73}$$

which can be further processed successfully by relation (56).

Before we proceed to any generalization of Lemma 18 we will use through Lemmas 19, 20 and 21 to investigate cases in which we attempt to use differential and contiguous function relations to achieve reduction to a desired hypergeometric function in the way we did in our previous example.

<u>Lemma 19.</u> Given the hypergeometric function

$${}_2F_1[\, m/n,\ b;\ k/l;\ z] \tag{74}$$

then under the following conditions the relation

$$2\left(\frac{m}{n} + x\right) = \frac{k}{l} + y \quad \text{for } m, k, x, y \in Z \text{ and } n, l \in Z - \{ 0 \} \tag{75}$$

cannot be satisfied

1. For any y and one of the following conditions

   a) $2l \nmid k$ and $n \mid m$

   b) $2l \mid k$ and $n \nmid m$

2. y odd, $n \mid m$ and $2l \mid k$

<u>Lemma 20.</u> Given the hypergeometric function (74) then under the following condition relation (75) is satisfied.

y is even and one of the following conditions holds

a) $2l \mid k$ and $n \mid m$

b) $2l = n$ and $2l = n \mid k-m$.

<u>Lemma 21.</u> Given the hypergeometric function (74) then relation (75) is satisfied or not according to the following criterion:

If y is even and if $2l \nmid k$ and $n \nmid m$ then (75) is satisfied depending on whether $2ln$ divides $kn - 2lm$ or not.

Lemmas 19, 20 and 21 can be easily proved by means of elementary number theory.

Hence, lemmas 19, 20 and 21 indicate when it is effective to use differentiation and/or contiguous function relations to a hypergeometric function in order to match a hypergeometric function of the type:

$$ {}_2F_1[\, a,\ a-1/2;\ 2a;\ z] \tag{76}$$

Of course similar criterions can be found for cases like

$$ {}_2F_1[\, a,\ a-1/2;\ 3a;\ z],\quad {}_2F_1[\, a,\ a-1/2;\ 4a;\ z] \tag{77}$$

and so on. However, we are not interested in such criterions for our present problems.

<u>Lemma 22.</u> Any hypergeometric function of the form

$$ {}_2F_1[\, a+m,\ -a+l;\ c;\ z] \quad m,\ l \in Z,\quad a,\ c,\ z \in C \tag{78}$$

can be represented in any of the following ways

$$ {}_2F_1[\, a'+n,\ -a';\ c;\ z] \tag{79} $$

$$ {}_2F_1[\, a'',\ -a''+n;\ c;\ z] \tag{80} $$

where $a', a'' \in C$ and $n \in Z$.

Lemma 23. For $m, n \in N$ $a, b, z \in C$ the following relations hold:

$$ {}_2F_1[\, a+m,\ b;\ 1/2+n;\ z] = \frac{(n-m+1/2)_m (1/2)_{n-m}}{(-1)^m (a)_m (1/2+n-m-b)_m (1/2-a)_{n-m} (1/2-b)_{n-m}} \tag{81} $$

$$ (1-z)^{1-a} \frac{d^m}{dz^m}[(1-z)^{n-b-1/2} \frac{d^{n-m}}{dz^{n-m}}[(1-z)^{a+b-1/2}\ {}_2F_1[\, a,\ b;\ 1/2;\ z]]] $$

where we assume here $m \leq n$

$$ {}_2F_1(a-m,\ b;\ 1/2-n;\ z] = \frac{1}{(1/2-n)_m (1/2-n+m)_{n-m}} z^{n+1/2} (1-z)^{m+1/2-b-n} \tag{82} $$

$$ \frac{d^m}{dz^m}[(1-z)^{n+b-1/2} \frac{d^{n-m}}{dz^{n-m}}[\, z^{-1/2}\ {}_2F_1[\, a,\ b;\ 1/2;\ z]]] $$

where here assume $m \leq n$

$$ {}_2F_1[\, a-m,\ b;\ 1/2+n;\ z] = \frac{(1/2)_n}{(1/2+n-a)_m (1/2-a)_n (1/2-b)_n} \tag{83} $$

$$ z^{1/2+a-n} (1-z)^{m+n+1/2-a-b} $$

$$ \frac{d^m}{dz^m}[\, z^{n+m-a-1/2} \frac{d^n}{dz^n}[(1-z)^{a+b-1/2}\ {}_2F_1[\, a,\ b;\ 1/2;\ z]]] $$

$$ {}_2F_1[\, a+m,\ b;\ 1/2-n;\ z] = \frac{1}{(a)_m (1/2-n)_n} z^{1-a} \frac{d^m}{dz^m}[\, z^{a+m+n-1/2} \tag{84} $$

$$\frac{d^n}{dz^n}\left[z^{-1/2}\, {}_2F_1[\, a,\ b;\ 1/2;\ z]\right]]$$

$${}_2F_1[\, a+m,\ b;\ 1/2+n;\ z] = \frac{(1/2)_n}{(-1)^m (a+m-n)_n (1/2-b)_n (a)_{m-n}} (1-z)^{1+n-a-m} \quad (85)$$

$$\frac{d^n}{dz^n}\left[(1-z)^{a+m-1}\, z^{1-a} \frac{d^{m-n}}{dz^{m-n}}\left[z^{a+m-n-1}\, {}_2F_1[\, a,\ b;\ 1/2;\ z]\right]\right]$$

where $n \leq m$

$${}_2F_1[\, a-m,\ b;\ 1/2-n;\ z] = \frac{1}{(1/2-n)_n (1/2-a)_{m-n}} z^{n+1/2}\, (1-z)^{1/2-b} \quad (86)$$

$$\frac{d^n}{dz^n}\left[\, z^a\, (1-z)^{m-a} \frac{d^{m-n}}{dz^{m-n}}\left[\, z^{m-n-a-1/2}\, (1-z)^{a+b-1/2}\, {}_2F_1[\, a,\ b;\ 1/2;\ z]\right]\right]$$

where we assume here $n \leq m$.

Proof Relation (81) can be proved from relations (31) and (30) of Chapter 2. Similarly, relation (82) can be proved from relations (32) and (28) of Chapter 2; relation (83) follows from (29) and (30) of Chapter 2; relation (84) from (27) and (28) of Chapter 2; relation (85) from (31) and (27) of Chapter 2; and finally relation (86) can be proved from relations (32) and (29) of Chapter 2.

Taking into account Lemma 22 and provided that $b = -a$ for relations (81) through (86), Lemma 23 constitutes the generalization of the relations (54) through (56) of Lemma 18 and the "limit-algorithm".

<u>Lemma 24.</u> For $x \in Z$, $y \in Z^+ \cup \{0\}$, $a, b, c, z \in C$ the following relations hold:

$$
{}_2F_1[\, a,\ b;\ c;\ z] = \frac{1}{(c)_y (c+y-a-x)_{x-y} (c-a-1/2+y-x)_{a-b+1/2+x}} \, z^{1-c} \, (1-z)^{y+c-b} \tag{87}
$$

$$
\frac{d^y}{dz^y}[\, z^{x+a} \, (1-z)^{-a} \, \frac{d^{x-y}}{dz^{x-y}}[\, z^{1/2+x-y}
$$

$$
\frac{d^{a-b+x+1/2}}{dz^{a-b+x+1/2}}[\, z^{c+y-1} \, (1-z)^{2a+2x-c-y+1/2} \, {}_2F_1[\, a+x,\ a+1/2+x;\ c+y;\ z]]]]
$$

where $a-b+x+1/2 \geq 0$, $x \geq y \geq 0$

$$
{}_2F_1[\, a,\ b;\ c;\ z] = \frac{1}{(c)_y (c+y-a-x)_{x-y} (a+1/2+x)_{b-a-x-1/2}} \tag{88}
$$

$$
z^{1-c} \, (1-z)^{y+c-b}
$$

$$
\frac{d^y}{dz^y}[\, z^{x+a} \, (1-z)^{-a} \, \frac{d^{x-y}}{dz^{x-y}}[\, z^{c-2a-x-1/2} \, (1-z)^{a+x+b-c-y}
$$

$$
\frac{d^{b-a-1/2-x}}{dz^{b-a-1/2-x}}[\, z^{b-1} \, {}_2F_1[\, a+x,\ a+x+1/2;\ c+y;\ z]]]]
$$

where $a-b+x+1/2 < 0$, $x \geq y \geq 0$

$$
{}_2F_1[\, a,\ b;\ c;\ z] = \frac{1}{(c)_x (c+x)_{y-x} (c-a-1/2+y-x)_{a-b+1/2+x}} \tag{89}
$$

$$
z^{1-c} \, (1-z)^{x+c-b} \, \frac{d^x}{dz^x}[(1-z)^{b-c} \, \frac{d^{y-x}}{dz^{y-x}}[z^{a+x+1/2} \, (1-z)^{c-a-b+y-x}
$$

$$
\frac{d^{a-b+x+1/2}}{dz^{a-b+x+1/2}}[z^{c+y-1} \, (1-z)^{2a+2x-c-y+1/2} \, {}_2F_1[\, a+x,\ a+x+1/2;\ c+y;\ z]]]]
$$

where $y > x \geq 0$, $a-b+x+1/2 \geq 0$

$$ {}_2F_1[a, b; c; z] = \frac{1}{(c)_x (c+x)_{y-x} (a+1/2+x)_{b-a-1/2-x}} \qquad (90) $$

$$ z^{1-c} (1-z)^{x+c-b} $$

$$ \frac{d^x}{dz^x}[(1-z)^{b-c} \frac{d^{y-x}}{dz^{y-x}}[z^{c+y-x-a-1/2} $$

$$ \frac{d^{b-a-1/2-x}}{dz^{b-a-1/2-x}}[z^{b-1} {}_2F_1[a+x+1/2, a+x; c+y; z]]]] $$

where $y > x \geq 0$ and $a-b+x+1/2 < 0$

$$ {}_2F_1[a, b; c; z] = \frac{1}{(a-w)_w (c)_y (c+y+w-a-1/2)_{a-w-b+1/2}} z^{1+w-a} \qquad (91) $$

$$ \frac{d^w}{dz^w}[z^{a-c} \frac{d^y}{dz^y}[z^{a-w+1/2} (1-z)^{w+c+y-a-b} $$

$$ \frac{d^{a-w-b+1/2}}{dz^{a-w-b+1/2}}[z^{c+y-b-1} (1-z)^{2a-2w-c-y+1/2} {}_2F_1[a-w, a+1/2-w; c+y; z]]]] $$

where $w := -x$, $a-w-b+1/2 \geq 0$, $x < 0$, $y \geq 0$

$$ {}_2F_1[a, b; c; z] = \frac{1}{(a-w)_w (c)_y (a+1/2-w)_{b-a-1/2+w}} z^{1+w-a} \qquad (92) $$

$$ \frac{d^w}{dz^w}[z^{a-c} \frac{d^y}{dz^y}[z^{c+y-a+w-1/2} $$

$$ \frac{d^{b-a-1/2+w}}{dz^{b-a-1/2+w}}[z^{b-1} {}_2F_1[a+1/2-w, a-w; c+y; z]]]] $$

where $w := -x$, $a-w-b+1/2 < 0$, $x < 0$, $y \geq 0$.

<u>Proof</u> Relation (87) is a consequence of relations (32) and (29) of Chapter 2; relation (88) of (32), (29) and (27) of Chapter 2; relation (89) of (32), (28) and (29) of Chapter 2; relation (90) of (32), (20) and (27) of Chapter

2; relation (91) of (27), (28) and (29) of Chapter 2; and finally relation (92) is implied from relations (27) and (28) of Chapter 2.

Lemma 24 through lemmas 19, 20 and 21, provide a generalization of the relations (58), (59), (60) and (61) of Lemma 18, and of course to our reduction-limit algorithm.

As it has been demonstrated, differentiation is one of our tools for reduction. A careful consideration of the different formulas involving differentiations shows that it might be the case that differentiation will be applied "dangerously" many times. That is, we might be differentiating an expression so many times that the continuously growing expression finally exceeds the limited storage capacity. It is our responsibility to provide formulas which utilize differentiation to a minimum; however, this of course does not quarantee that the aforementioned problem will not arise.

If we inspect the differentiation relations (26) through (33) of Chapter 2, we conclude that in order to increment a parameter of a hypergeometric function by one, we should diffferentiate the hypergeometric function once, and so on.

We establish the "lower bound" for the number of differentiations that must be applied to the hypergeometric function

$$ {}_2F_1[\, a,\ b;\ c;\ z] \qquad (93)$$

in its reduction to the hypergeometric function

$$ {}_2F_1[\, a+l,\ b+m;\ c+n;\ z] \qquad (94)$$

where l, m, n integer numbers, to be:

1. $|\max\{l,m,n\} - \min\{l,m,n\}|$, if there exist at least two integers among l, m and n with different signs.

2. $\max\{|l|, |m|, |n|\}$, if all of them have the same sign.

Acceptance of the above lower bound is further supported by the fact that no linear transformation of expression (93) can provide a better lower bound.

It is not difficult to see now that all of the formulas in Lemma 23 have accomplished optimal bounds in the number of differentiations applied. This is not true for the formulas of Lemma 24, it can be accomplished if we divide the present cases of Lemma 24 into further subcases (a rather routine and boring task by now).

The following theorem summarizes our results of the above lemmas.

Theorem 1. The following hypergeometric functions reduce to Binomial and/or Elementary Functions

1. ${}_2F_1[\, a+l,\ -a+m;\ 1/2+n;\ z]$ (95)
2. ${}_2F_1[\, 1/2-a+l,\ 1/2+a+m;\ 1/2+n;\ z]$ (96)
3. ${}_2F_1[\, 1/2-a+l,\ -a+m;\ 1/2+n;\ z]$ (97)
4. ${}_2F_1[\, a+l,\ a+m+1/2;\ n+1/2;\ z]$ (98)
5. ${}_2F_1[\, a+m_1/n_1,\ a+r+1/2;\ 2a+k_1/l_1;\ z]$ (99)

for $a \in C$, $l, m, n, r, m_1, k_1 \in Z$, $n_1, l_1 \in Z - \{0\}$

and under the following conditions

$$2l_1 \mid k_1 \quad \text{and} \quad n_1 \mid m_1$$

or $2l_1 = n_1$ and $n_1 \mid k_1 - m_1$

Proof This is a consequence of Lemmas 19 through 24.

The above lemmas and theorem are materialized in the following final "reduction-limit" algorithm.

Algorithm $_2F_1$-RL

Given the hypergeometric function

$$_2F_1[\text{ alpha, beta; gamma; arg}] \tag{100}$$

then

Step 1. Standardize the parameters of the hypergeometric function by using the Gauss-Euler transformations (160) and (161).

Step 2. If alpha = a + l, beta = -a + m and

gamma = 1/2+n, where a $\in$ C, l, m, n $\in$ Z

then go to step 4

Step 3. If alpha - beta + 1/2 is an integer number

then go to step 7

else go to step 12

Step 4. Call algorithm II and standardize the parameters of the hypergeometric function. Thus the quantities a, l, m, and n will be calculated and the hypergeometric function put into the form

$$_2F_1[\text{ alpha'+m', -alpha'; 1/2+n'; z}] \tag{101}$$

<u>Step 5.</u> Call algorithm I to process the hypergeometric function

$${}_2F_1[\text{ alpha'}, \text{-alpha'}; 1/2; z] \qquad (102)$$

<u>Step 6.</u> Call algorithm III and dispatch to it the result of step 5 as well as the quantities m, n, then go to step 11.

<u>Step 7.</u> Call algorithm IV, test the parameters and calculate the quantities x and y.

<u>Step 8.</u> If "test" in step 7 fails

<u>then</u> go to step 12

<u>else</u> set alpha' to the value of "alpha + x"

<u>Step 9.</u> Call algorithm I to process the hypergeometric function

$${}_2F_1[\text{ alpha'}, \text{alpha'}-1/2; \text{2alpha'}; z] \qquad (103)$$

<u>Step 10.</u> Call algorithm V and dispatch to it the result of step 9. as well as the quantities x and y.

<u>Step 11.</u> Return result.

<u>Step 12.</u> Return fail.

The following algoritm is actually the implementation of lemma 22.

<u>Algorithm II.</u> Given

$${}_2F_1[\text{ a+m}, \text{-a+n}; c; z] \qquad (104)$$

where m, n are integer numbers, then

<u>Step 1.</u> If the product mn is a nonnegative number

then go to Step 3

else go to Step 2

Step 2. If $|m| > |n|$ then go to Step 4

Step 3. Set a' equal to a+m and return

$$_2F_1[\ -a'+(m+n),\ -(-a');\ c;\ z] \tag{105}$$

Step 4. Set a' equal to a-n and return

$$_2F_1[\ a'+(m+n),\ -a';\ c;\ z] \tag{106}$$

The main function of Algorithm II is to standardize the parameters of the hypergeometric function and minimize the number of differentiations that must be performed at a later stage of the limit reduction algorithm. Hence, for example the hypergeometric function

$$_2F_1[\ a+1000,\ -a-950;\ c;\ z] \tag{107}$$

will need at least 1950 differentiations to be reduced to the hypergeometric function

$$_2F_1[\ a,\ -a;\ c;\ z] \tag{108}$$

However, if algorithm II is initially performed to expression (107) it will reduce it to

$$_2F_1[\ a'+50,\ -a';\ c;\ z] \tag{109}$$

where $a' := a + 700$

thus, reducing the number of the required differentiations to 50.

The following two algorithms called by the major algorithm $_2F_1$-RL are the implementations of the ideas presented in lemmas 19, 20, 21, 23 and 24.

<u>Algorithm III.</u> Given the hypergeometric function

$${}_2F_1[\ a+m,\ b;\ 1/2+n;\ z] \tag{110}$$

then

<u>Step 1.</u> If m is positive <u>then</u> go to step 5

<u>Step 2.</u> If n is negative <u>then</u> go to step 3

<u>else</u> go to step 4

<u>Step 3.</u> If $|m| \le |n|$ <u>then</u> apply formula (82) and return

<u>else</u> apply formula (86) and return

<u>Step 4.</u> Apply formula (83) and return.

<u>Step 5.</u> If n positive <u>then</u> go to step 6

<u>else</u> apply formula (84) and return.

<u>Step 6.</u> If $m \le n$ <u>then</u> apply formula (81) and return

<u>else</u> apply formula (85) and return.

<u>Algorithm IV.</u> Given

$${}_2F_1[\ alpha,\ beta;\ gamma;\ z] \tag{111}$$

then

<u>Step 1.</u> Separate the numeric from the nonumeric part in the parameters of the hypergeometric function (111)

<u>Step 2.</u> If the nonumeric part satisfies the condition $2a = c$

<u>then</u> go to step 3

else go to step 10.

Step 3. Standardize the numeric parts as follows:

Numeric part of alpha = m/n

Numeric part of beta = m/n + r - 1/2

Numeric part of gamma = k/l

and in such a fashion that all fractions m/n, k/l are irreducible.

Step 4. If 2l = n *then* go to step 5

else go to step 6

Step 5. If n | k-m *then* go to step 11

Step 6. If 2l | k *then* go to step 7

else go to step 8

Step 7. If n | m *then* go to step 11

else go to step 10

Step 8. If n | m *then* go to step 10

Step 9. If 2ln | kn-2lm *then* go to step 11

Step 10. Return "not satisfied".

Step 11. For y = 0, 2, 4,... find the first integer number x satisfying the relation

$$x = \frac{k/l-2m/n+y}{2}$$

and return the values of x and y.

Algorithm V. Given ${}_2F_1[a, b; c; z]$ and x, y values, then

Step 1. If $x \geq 0$ then go to step 2

else go to step 5

Step 2. If $x \geq y$ then go to step 3

else go to step 4

Step 3. If $a-b+x+1/2 \geq 0$ then apply formula (87) and return

else apply formula (88) and return

Step 4. If $a-b+x+1/2 \geq 0$ then apply formula (89) and return

else apply formula (90) and return

Step 5. Set $w := -x$

Step 6. If $a-w-b+1/2 \geq 0$ then apply formula (91) and return

else apply formula (92) and return.

The algorithms presented are a demonstration of the main flow of control. We have tried to minimize the details which contribute to the efficiency of an implementation. Some of them are mentioned but we avoid repeating them in other instances since they are obviously implied.

We conclude the "limit-reduction of the hypergeometric functions" section by illustrating the function of ${}_2F_1$-RL algorithm in an example.

<u>Example</u> Given the hypergeometric function

$$ {}_2F_1[\ a+5/6,\ a+7/3;\ 2a+23/3;\ 4z(1-z)] \qquad (112)$$

then

<u>Algorithm $_2F_1$-RL</u>: Step 3 is satisfied and at step 7 algorithm IV is called.

<u>Algorithm IV</u>: Given the hypergeometric function (112), step 2 is satisfied and at step 3 we have

$$ m/n = 5/6 \quad r = -2 \quad k/l = 23/3 \qquad (113)$$

Steps 4 and 5 are satisfied, hence step 11 provides

$$ x = 3 \quad y = 0 \qquad (114)$$

and return to Algorithm $_2F_1$-RL.

At step 8, we set alpha':= a + 23/6. At step 9 we call algorithm I with input

$$ {}_2F_1[\ a+23/6,\ a+10/3;\ 2a+23/3;\ z] \qquad (115)$$

<u>Algorithm I</u>

<u>Step 1</u> It is true that

$$ \lim_{c\to 0} {}_2F_1[\ a+23/6-c/2,\ (c+2a+20/3)/2;\ 2a+23/3;\ 4z(1-z)] =$$

$$ = {}_2F_1[\ a+23/6,\ a+10/3;\ 2a+23/3;\ 4z(1-z)] \qquad (116)$$

<u>Step 2</u> It is also true that

$$ {}_2F_1[\ a+23/6-c/2,\ (c+2a+20/3)/2;\ 2a+23/3;\ 4z(1-z)] = \qquad (117)$$

$$ = (1-z)^{-2a-20/3}\ {}_2F_1[\ c,\ 1-c;\ 2a+23/3;\ z]$$

<u>Step 3</u> Hence, we can get

$$\lim_{c \to 0} (1-z)^{-2a-20/3} \; {}_2F_1[\, c,\ 1-c;\ 2a+23/3;\ z] = \tag{118}$$

$$= (1-z)^{-20/3-2a} \; {}_2F_1[\, 0,\ 1;\ 2a+23/3;\ z]$$

<u>Step 4</u> For the right hand side hypergeometric function of relation (118), the "reduction algorithm" provides 1. Therefore algorithm I returns back:

$$(1-z)^{-20/3-2a} \tag{119}$$

Given the results (114) and (119) algorithm ${}_2F_1$-RL at step 10 calls algorithm V.

<u>Algorithm V</u> Steps 1 through 3 of algorithm V are satisfied, therefore formula (87) ultimately provides the result (120) which is also the final result of our principal algorithm

$$-16/27\; 4^{-a-17/6}\; (1-z)^{-a-14/6}\; z^{a+7/3}\; (1-4(1-z)z)^{9/2} \tag{120}$$

$$[(20736a^4 + 442368a^3 + 3502656a^2 + 12180288a + 15662080)\; z^4$$

$$-(72576a^4 + 1413504a^3 + 10227168a^2 + 32552448a + 38420480)\; z^3$$

$$+(93312a^4 + 1640736a^3 + 10726992a^2 + 30898800a + 33078528)\; z^2$$

$$-(51840a^4 + 810432a^3 + 4715064a^2 + 12101628a + 11563216)\; z$$

$$+ 10368a^4 + 140832a^3 + 712584a^2 + 1591998a + 1325303]$$

3.2.3.2. LOGARITHMIC AND OTHER ALGEBRAIC CASES

Our next lemmas and theorems are concerned with logarithmic and algebraic functions.

Lemma 25. The following relation is true for any complex z

$$_2F_1[\ 1,\ 1;\ 2;\ z] = -\ z^{-1}\ \log(1-z) \tag{121}$$

Lemma 26. The following relation

$$_2F_1[\ n+1,\ n+m+1;\ n+m+l+2;\ z] = \tag{122}$$

$$\frac{(-1)^m(n+m+l+1)!}{l!n!(n+m)!(m+l)!}\ \frac{d^{n+m}}{dz^{n+m}}[(1-z)^{m+l}\ \frac{d^l}{dz^l}\,{}_2F_1[\ 1,\ 1;\ 2;\ z]]$$

holds for any $l, m, n = 0, 1, 2, \ldots$

Proof Relation (122) is implied from relations (26) and (32) of Chapter 2.

Theorem 2. For any integer value of the parameters a, b and positive integer values of the parameter c the hypergeometric function $_2F_1[\ a,\ b;\ c;\ z]$ reduces to Elementary and/or Binomial functions.

Proof This is a consequence of lemma 25 above and lemmas 1 through 5 of our general reduction part.

Lemma 27. The following relation holds

$$_2F_1[\ 1/2,\ 1;\ 2;\ 4z(1-z)] = (1-z)^{-1} \tag{123}$$

for $|z| \leq 1/2,\ \ |z(1-z)| \leq 1/4$

Lemma 28. The following relation holds

$$ {}_2F_1[\, 1/2+l,\ 1+m;\ 2+n;\ z] = \frac{(2)_m (2+m)_{n-m}}{(-1)_{n-m} (1/2)_m (1/2+m)_{n-m} (1)_m (1)n_{n-m} (1/2+n)_{l-m}} \tag{124} $$

$$ z^{1/2-n} \frac{d^{l-n}}{dz^{l-n}} [z^{n-1/2} \ (1-z)^{1/2-m} \frac{d^{n-m}}{dz^{n-m}} [(1-z)^{n-1/2} \frac{d^m}{dz^m}\ {}_2F_1[\, 1/2,\ 1;\ 2;\ z]]] $$

Proof Relation (124) is implied from relations (26), (31) and (27) of Chapter 2.

Our next theorem summarizes our reductions so far accomplished for hypergeometric functions with numeric parameters and leading to Elementary and algebraic functions.

Theorem 3. The hypergeometric function

$$ {}_2F_1[\, a,\ b;\ c;\ z] \tag{125} $$

is reducible to some Elementary and/or algebraic functions for any

$$ a,\ b \in \{x \mid x = n \ \text{ or } \ x = n/2,\ \ n \in Z\} $$

$$ c \in \{x \mid x = m \ \text{ or } \ x = n/2 \ \ n \in Z,\ m \in Z^{+}\} $$

Proof The proof follows from lemma 26 above, theorems 1 and 2 as well as lemmas 1 through 5 of the general reduction part and the use of Gauss-Euler transformations (160) and (161).

3.2.3.3. LEGENDRE FUNCTION REDUCTIONS

We next examine cases where a hypergeometric function is reducible to Legendre functions.

Lemma 29. A hypergeometric function

$$ {}_2F_1[\,a,\ b;\ c;\ z] \qquad (126)$$

is reducible to a Legendre function if two of the numbers $1-c$, $\pm(a-b)$, $\pm(c-a-b)$ are equal to each other or one of them equals $\pm 1/2$.

Proof This is a consequence of lemma 17 and relations (61) and (62) of Chapter 3.

Our next step again is to generalize Lemma 29 by using contiguity and/or differentiation.

Lemma 30. Given the hypergeometric function (126) such that

$$a+b = 1+m, \quad m \in Z^+ \qquad (127)$$

then the following relation holds

$$ {}_2F_1[\,a,\ b;\ c;\ z] \qquad (128)$$

$$= (-1)^m (1-z)^{-m}$$

$$[\binom{m}{0} \frac{(c-b)(c-b+1)\ldots(c-b+(m-2))(c-b+(m-1))}{(b-a-(m-1))(b-a-(m-2))\ldots(b-a-1)(b-a)} {}_2F_1[\,a,\ b-m;\ c;\ z]$$

$$- \binom{m}{1} \frac{(c-a)(c-b)(c-b+1)\ldots(c-b+(m-3))(c-b+(m-2))}{(b-a-(m-1))(b-a-(m-3))\ldots(b-a)(b-a+1)} {}_2F_1[\,a-1,\ b-(m-1);\ c;\ z]$$

$$+ \binom{m}{2} \frac{(c-a)(c-a+1)(c-b)(c-b+1)\ldots(c-b+(m-3))}{(b-a-(m-2))(b-a-(m-3))\ldots(b-a+1)(b-a+2)} \; {}_2F_1[\, a-2,\; b-(m-2);\; c;\; z]$$

... ...

$$+ (-1)^m \binom{m}{m} \frac{(c-a)(c-a+1)\ldots(c-a+(m-2))(c-a+(m-1))}{(b-a)(b-a+1)\ldots(b-a+(m-2))(b-a+(m-1))} \; {}_2F_1[\, a-m,\; b;\; c;\; z]\;]$$

<u>Proof</u> Relation (128) can be proved inductively on m, by using relation (12) of Chapter 2.

Relation (128) has meaning whenever quantities a and b are not integer numbers or rationals of the same denominator. Actually, relation (128) is of benefit whenever quantities a and b are complex numbers (or generally whenever a and b contain symbolic quantities). Relation (128) obviously reduces a hypergeometric function to a sum of hypergeometrics such that lemma 28 holds for all of them.

<u>Lemma 31.</u> Given the hypergeometric function (126) such that

$$a+b = 1+m, \quad m \in Z^{-}-\{-1\} \tag{129}$$

then the following relation holds

$${}_2F_1[\, a,\; b;\; c;\; z] \tag{130}$$

$$= \binom{n}{0} (b-a-n) \frac{a(a+1)\ldots(a+(n-2))(a+(n-1))}{(b-a-n)(b-a-(n-1))\ldots(b-a-1)(b-a)} \; {}_2F_1[\, a+n,\; b;\; c;\; z]$$

$$- \binom{n}{1} (b-a-n+2) \frac{a(a+1)\ldots(a+(n-2))b}{(b-a-(n-1))(b-a-(n-2))\ldots(b-a)(b-a+1)} \; {}_2F_1[a+n-1, b+1;\; c;\; z]$$

$$+\binom{n}{2}(b-a-n+4)\frac{a(a+1)\ldots(a+(n-4))b(b+1)}{(b-a-(n-2))(b-a-(n-3))\ldots(b-a+1)(b-a+2)}\,{}_2F_1[a+n-2,b+2;c;z]$$

... ...

$$+\ (-1)^n\binom{n}{n}(b-a+n)\frac{b(b+1)\ldots(b+(n-1))}{(b-a)(b-a+1)\ldots(b-a+(n-1))(b-a+n)}\,{}_2F_1[a,b+n;\ c;\ z]$$

where $n := -m$.

Proof Relation (130) can be proved by induction on n using the contiguous function relation (7) of Chapter 2.

Relation (130) reduces the hypergeometric function (126) that satisfies a condition of type (129), into a sum of hypergeometrics that meets the conditions of lemma 25.

We should notice that in order that a and b satisfy relation (130), both should be either integers, rational numbers of the same denominator, coplex, or both contain symbols. If both are integers then at least one must be negative which implies that lemma 26 applies reducing the hypergeometric function to a polynomial. It is easy to see that in any of the rest of the cases relation (130) is applied without any restrictions.

Lemma 32. Given the hypergeometric function (126) such that

$$a+b = 2c+m-1,\quad m \in Z^+ \tag{131}$$

then relation (128) reduces (126) into a sum of hypergeometric functions such that the conditions of lemma 28 are satisfied.

<u>Lemma 33.</u> Given the hypergeometric function (126) such that

$$a+b = 2c+m-1, \quad m \in Z^- \tag{132}$$

then relation (130) reduces (126) into a sum of hypergeometric functions such that lemma's 28 conditions are satisfied.

<u>Lemma 34.</u> Given the hypergeometric function (126) such that

$$a-b = \pm(1-c)+m, \quad m \in Z^+ \tag{133}$$

then the following relation holds

$$
\begin{aligned}
{}_2F_1[\,a,\ b;\ c;\ z] & \qquad\qquad (134) \\
= \binom{m}{0} & (c-b-a-m)\frac{b(b+1)(b+2)\dots(b+(m-1))}{(c-b-a-m)(c-b-a-m+1)\dots(c-b-a)}(z-1)^m\ {}_2F_1[\,a,\ b+m;\ c;\ z] \\
+ \binom{m}{1} & (c-b-a-(m-2))\ \frac{b(b+1)\dots(b+(m-2))(c-a)}{(c-b-a-(m-1))(c-b-a-(m-2))\dots(c-b-a+1)}\ (z-1)^{m-1} \\
& \qquad {}_2F_1[a-1,\ b+(m-1);\ c;\ z] \\
\dots & \quad \dots \quad \dots \\
+ \binom{m}{m} & (c-b-a+m)\ \frac{(c-a)(c-a+1)\dots(c-a+(m-1))}{(c-b-a)(c-b-a+1)\dots(c-b-a+m)}\ {}_2F_1[\,a-m,\ b;\ c;\ z]
\end{aligned}
$$

<u>Proof</u> Relation (134) can be proved by induction on m, given the relation (11) of Chapter 2.

Relation (134) holds as long as quantities a, b and c are not integer numbers; clearly it reduces the appropriate hypergeometric function into others which fall within the capabilities of lemma 29.

<u>Lemma 35.</u> Given the hypergeometric function (126) such that

$$a-b = \pm(1-c)+m, \quad m \in Z^- \tag{135}$$

then the formula which is produced from formula (134) by an application of a cyclic permutation to the quantities a and b, holds.

Lemma 36. Given the hypergeometric function (126) such that

$$a-b = 1/2+m, \quad m \in Z^+ \tag{136}$$

then relation (134) holds and reduces (126) to a sum of hypergeometrics elligible for lemma 29.

Lemma 37. Given the hypergeometric function (126) such that

$$a-b = 1/2+m, \quad m \in Z^- \tag{137}$$

then relation (134) holds by first applying a cyclic permutation of the quantities a and b.

Lemma 38. Given the hypergeometric function (126) such that

$$a+b = 1/2+c+m, \quad m \in Z^+ \tag{138}$$

then relation (128) holds and reduces (126) into a sum of elligible for lemma 29 hypergeometrics.

Lemma 39. Given the hypergeometric function (126) such that

$$a+b = 1/2+c+m, \quad m \in Z^- \tag{139}$$

then relation (130) applies and reduces (126) to a sum of hypergeometric functions such that relation $a+b = 1/2+c$ is satisfied.

In generalizing the conditions of lemma 29 to accomplish reduction of Legendre functions, we have been successful so far, by utilizing as our main tool "contiguity". However, this is not possible for the rest of the cases we are going to deal with. Therefore, we mobilize our last resort: "differentiation". Unfortunately, despite our many experiments with "differentiation" in these remaining cases, we have been persuaded that general formulas cannot be provided. However, computational methods are possible and are described next.

Lemma 40. There is an algorithm such that the hypergeometric function (126) with

$$c = 1/2+m, \quad m \in Z^+ \tag{140}$$

reduces to a sum of Legendre functions.

Proof This is immediate from relation (26) of Chapter 2, lemma 29 and the following relation:

$$\frac{dP_{v,\mu}(z)}{dz} = vz\, P_{v,\mu}(z) - (v+\mu)\, P_{v-1,\mu}(z) \tag{141}$$

Lemma 41. There is an algorithm such that the hypergeometric function (126) with

$$c = 1/2+m, \quad m \in Z^- \tag{142}$$

reduces to a sum of Legendre functions.

Proof This is a result of lemma 29, relation (30) of Chapter 2 and relation (141).

Lemma 42. There is an algorithm such that the hypergeometric function (126) with

$$2b = c+m, \quad m \in Z \tag{143}$$
$$\text{(and symmetrically } 2a = c+m)$$

reduces to a sum of Legendre functions.

Proof Similar remarks to the ones presented in lemmas 40 and 41 apply here too.

We conclude our discussion of reduction to Legendre functions with the following theorem which is a generalization of our lemma 29.

<u>Theorem 4.</u> A hypergeometric function (126) is reducible to:

a) A Legendre function if two of the numbers $1-c$, $\pm(a-b)$ and $\pm(c-a-b)$ are equal to each other or one of them equals $\pm 1/2$,

b) A sum of Legendre functions, if one of the following conditions hold:

1. $a+b = 1+m$, $m \in Z^+$ and $a, b \in C - Q$
2. $a+b = 1+m$, $m \in Z^- - \{-1\}$ and $a, b \in C - Z$
3. $a+b = 2c+m-1$, $m \in Z^+$ and $a, b \in C - Q$
4. $a+b = 2c+m-1$, $m \in Z^- - \{-1\}$ and $a, b \in C - Z$
5. $a-b = \pm(1-c)+m$, $m \in Z - \{0\}$ and $a, b, c \in C - Z$
6. $a-b = 1/2+m$, $m \in Z - \{0\}$ and $a, b, c \in C - Z$
7. $a+b = 1/2+c+m$, $m \in Z^+$ and $a, b \in C - Q$
8. $a+b = 1/2+c+m$, $m \in Z^- - \{-1\}$ and $a, b \in C - Z$
9. $c = 1/2+m$, $m \in Z$ and $c \in C$
10. $2b = c+m$, symmetrically $2a = c+m$, $m \in Z$ and $a, b, c \in C$

<u>Proof</u> This is a consequence of lemmas 29 through 42 of the present section.

3.2.3.4. THE ORTHOGONAL POLYNOMIALS OF JACOBI GEGENBAUER LEGENDRE AND TCHEBICHEF

The Orthogonal Polynomials of Jacobi, Gegenbauer, Legendre and Tchebichef belong to the set of hypergeometric functions. Every hypergeometric function

$$ {}_2F_1[\, a,\ b;\ c;\ z] \tag{144}$$

where a or b is a negative integer reduces to some of the above mentioned polynomials. Under certain conditions, it is also possible to reduce (144) into some of the above Orthogonal Polynomials where a or b is some other arbitrary quantity. To accomplish the latest case, we use linear, cubic as well as other of higher degree transformations.

Lemma 43. The following relations hold:

$$T_n(x) = {}_2F_1[\, -n,\ n;\ 1/2;\ 1/2-x/2] \tag{145}$$

$$U_n(x) = (n+1)\ {}_2F_1[\, -n,\ n+1;\ 3/2;\ 1/2-x/2] \tag{146}$$

$$P_n(x) = {}_2F_1[\, -n,\ n+1;\ 1;\ 1/2-x/2] \tag{147}$$

$$C_{n,v}(x) = \frac{(2v)_n}{n!}\ {}_2F_1[\, -n,\ n+2v;\ v+1/2;\ 1/2-x/2] \tag{148}$$

$$P_{n,\alpha,\beta}(x) = \binom{n+\alpha}{n}\ {}_2F_1[\, -n,\ n+\alpha+\beta+1;\ \alpha+1;\ 1/2-x/2] \tag{149}$$

where $n \in N$, $\alpha, \beta \in C$ and where $T_n(x)$ and $U_n(x)$ are Tchebichef polynomials, $P_n(x)$ a Legendre polynomial, $C_{n,v}(x)$ a Gegenbauer polynomial and $P_{n,\alpha,\beta}(x)$ a Jacobi polynomial.

Of course, there exist equivalent hypergeometric function relations in addition to those listed above (145) through (149). However, as lemma 43 implicitly communicates, we have established that a necessary feature of the "standard" form of our Orthogonal Polynomials is that they should have a negative integer number as a member of their "L_l list". We later on provide rules so that equivalent hypergeometric functions are reduced to the standard ones: (145) through (149).

<u>Lemma 44.</u> The following relations hold:

$$T_n(x) = n/2\ C_{n,0}(x) \tag{150}$$

$$T_n(x) = \frac{n!}{(1/2)_n} P_{n,-1/2,1/2}(x) \tag{151}$$

$$U_n(x) = C_{1,n}(x) \tag{152}$$

$$U_n(x) = \frac{(n+1)!}{2(1/2)_{n+1}} P_{n,1/2,1/2}(x) \tag{153}$$

$$P_n(x) = C_{n,1/2}(x) \tag{154}$$

$$P_n(x) = P_{n,0,0}(x) \tag{155}$$

$$C_{n,v}(x) = \frac{(2v)_n}{(v+1/2)_n} P_{n,v-1/2,v-1/2}(x) \tag{156}$$

<u>Proof</u> Relations (150) through (156) are a consequence of relations (145) through (149).

Hence, Legendre and Tchebichef polynomials are particular cases of Gegenbauer polynomials and in their turn Gegenbauer polynomials are particular cases of the polynomials of Jacobi. Furthermore, Jacobi polynomials cover the whole spectrum of polynomials that belong to the hypergeometric function set.

Almost in every case so far the linear transformations - Gauss Euleur transformations - have been used to "standardize" and transform a hypergeometric function to a canonical form which is possibly further reducible to some Elementary or Special Function. Orthogonal Polynomials are not an exception. Linear transformations are utilized and in addition to them quadratic, cubic as well as transformations of fourth and sixth degree are also put into action. Hence, we will next concetrate on the utilization of the available transformations of a hypergeometric function.

The following lemma, similar to the lemma 17 of the Limit-Reduction subsection, provides the criterion for accepting a hypergeometric function as elligible for third, fourth or sixth degree transformations.

<u>Lemma 45.</u> A cubic, quadratic or sixth degree transformation of the hypergeometric function

$$_2F_1[\, a,\, b;\, c;\, z] \tag{157}$$

exists if and only if either

$$1-c = \pm(a-b) = \pm(c-a-b) \tag{158}$$

or if two of the numbers

$$\pm(1-c),\ \pm(a-b),\ \pm(c-a-b) \tag{159}$$

are equal to 1/3 (cubic), 1/4 (quartic) or 1/6 (sixth degree) correspondingly.

For an extensive list of higher degree transformations look the Appendix 2 and reference [15].

<u>Lemma 46.</u> The hypergeometric function (144) reduces to an Orthogonal Polynomial of lemma 44, if a or b or c-a or c-b are negative integers.

Proof This is a consequence of the following linear transformations, also called Gauss-Euleur transformations.

$$ {}_2F_1[\, a,\ b;\ c;\ z] = (1-z)^{c-a-b}\ {}_2F_1[\, c-a,\ c-b;\ c;\ z] \qquad (160)$$

$$ {}_2F_1[\, a,\ b;\ c;\ z] = (1-z)^{-a}\ {}_2F_1[\, a,\ c-b;\ c;\ \frac{z}{z-1}] \qquad (161)$$

Lemma 47. For any $a = -n/3$, $n \in N$, the following hypergeometric functions reduce to some Orthogonal Polynomials of our lemma 44.

$$\begin{array}{ll} {}_2F_1[\, a,\ a+1/3;\ 1/2;\ z] & {}_2F_1[\, a,\ 1/6-a;\ 1/2;\ z] \\ {}_2F_1[\, 1/2-a,\ 1/6-a;\ 1/2;\ z] & {}_2F_1[\, 1/2-a,\ 1/3+a;\ 1/2;\ z] \\ {}_2F_1[\, a,\ a+1/2;\ 2a+5/6;\ z] & {}_2F_1[\, a,\ a+1/3;\ 2a+5/6;\ z] \\ {}_2F_1[\, a-5/6,\ a+1/3;\ 2a+5/6;\ z] & {}_2F_1[\, a-5/6,\ a+1/2;\ 2a+5/6;\ z] \end{array} \qquad (162)$$

Proof This is a consequence of the following two cubic transformations:

$$ {}_2F_1[\, 3a,\ 3a+1/2;\ 4a+2/3;\ z] \qquad (163)$$

$$ = (1 - \frac{3z}{4})^{-3a}\ {}_2F_1[\, a,\ a+1/3;\ 2a+5/6;\ \frac{-27z^2(1-z)}{(3z-4)^3}]$$

$$ {}_2F_1[\, 3a,\ 1/3-a;\ 1/2;\ z] \qquad (164)$$

$$ = (1-z)^{-a}\ {}_2F_1[\, a,\ 1/6-a;\ 1/2;\ \frac{(9-8z)^2 z}{27(1-z)}]$$

as well as the Gauss-Euler relations (160) and (161).

Lemma 48. For any $a = -n/3$, $n \in N$, the following hypergeometric functions reduce to Orthogonal Polynomials mentioned in Lemma 44.

$$
\begin{array}{ll}
{}_2F_1[\, a,\ a+1/2;\ 2/3;\ z] & {}_2F_1[\, a,\ 1/6-a;\ 2/3;\ z] \\
{}_2F_1[\, 2/3-a,\ a+1/2;\ 2/3;\ z] & {}_2F_1[\, 2/3-a,\ 1/6-a;\ 2/3;\ z]
\end{array} \qquad (165)
$$

Proof This is a consequence of the following qubic transformation

$$
{}_2F_1[\, a,\ 1/6-a;\ 2/3;\ x] \qquad (166)
$$

$$
= \frac{\Gamma(1/2-a)\Gamma(a+1/3)}{3\pi^{1/2}\Gamma(1/3)} [(1-t_1)^a\ {}_2F_1[\, 3a,\ 1/3-a;\ 1/2;\ t_1]
$$

$$
+ (1-t_2)^a\ {}_2F_1[\, 3a,\ 1/3-a;\ 1/2;\ t_2]
$$

$$
+ (1-t_3)^a\ {}_2Ff_1[\, 3a,\ 1/3-a;\ 1/2;\ t_3]]
$$

where t_1, t_2 and t_3 are roots of the following cubic equation

$$
(3-4t)^3 - 27(1-t)x = 0 \qquad (167)
$$

as well as the Gauss Euleur transformations.

Lemma 49. For any $a = -n/4$, $n \in N$, the following hypergeometric functions reduce to some Orthogonal Polynomials of our lemma 44.

$$
\begin{array}{ll}
{}_2F_1[\, a,\ 1/6-a;\ 2/3;\ z] & {}_2F_1[\, 2/3-a,\ 1/6-a;\ 2/3;\ z] \\
{}_2F_1[\, a,\ a+1/2;\ 2/3;\ z] & {}_2F_1[\, 2/3-a,\ a+1/2;\ 2/3;\ z] \\
{}_2F_1[\, a,\ a+1/4;\ 2a+3/4;\ z] & {}_2F_1[\, a+3/4,\ a+1/4;\ 2a+3/4;\ z] \\
{}_2F_1[\, a,\ a+1/2;\ 2a+3/4;\ z] & {}_2F_1[\, a+3/4,\ a+1/2;\ 2a+3/4;\ z]
\end{array} \qquad (168)
$$

Proof This is a consequence of the following two quartic transformations:

$$
{}_2F_1[\, 4a,\ 1/2-2a;\ 2/3;\ z] \qquad (169)
$$

$$
= (1-z)^{-a}\ {}_2F_1[\, a,\ 1/6-a;\ 2/3;\ \frac{(8-9z)^3 z}{64(1-z)}]
$$

$$
{}_2F_1[\, 4a,\ 2a+1/4;\ 2a+3/4;\ z] \qquad (170)
$$

$$= (1+z)^{-4a} {}_2F_1[a, a+1/4; 2a+3/4; \frac{16z(1-z)^2}{(1+z)^4}]$$

as well as the Gauss-Euler transformations (160) and (161).

Lemma 50. For $a = -n/4$, $n \in N$, the following hypergeometric functions reduce to Orthogonal Polynomials of lemma 44.

$$\begin{array}{ll} {}_2F_1[a, a+1/4; 1/2; z] & {}_2F_1[a, 1/4-a; 1/2; z] \\ {}_2F_1[1/2-a, 1/4-a; 1/2; z] & {}_2F_1[1/2-a, a+1/4; 1/2; z] \\ {}_2F_1[a, a+1/2; 3/4; z] & {}_2F_1[a, 1/4-a; 3/4; z] \\ {}_2F_1[3/4-a, 1/4-a; 3/4; z] & {}_2F_1[3/4-a, a+1/2; 3/4; z] \end{array} \qquad (171)$$

Proof This is a consequence of the following quartic transformations

$${}_2F_1[a, 1/4-a; 1/2; x] \qquad (172)$$

$$= \frac{16^a\Gamma(a+1/2)\Gamma(a+3/4)}{2\pi^{1/2}\Gamma(2a+3/4)}$$

$$[t_1^a(1-t_1)^{2a} {}_2F_1[4a, 2a+1/4; 2a+3/4; t_1]$$

$$+ t_2^a(1-t_2)^{2a} {}_2F_1[4a, 2a+1/4; 2a+3/4; t_2]]$$

where t_1, t_2 designate the two roots of the equation

$$(t^2-6t+1)^2 + 16t(1-t)^2x = 0 \qquad (173)$$

which are equal to 3-2sqrt(2) for $x = 0$.

$${}_2F_1[a, 1/4-a; 3/4; x] \qquad (174)$$

$$= \frac{16^a\Gamma(a+1/4)\Gamma(a+3/4)}{4\Gamma(1/4)\Gamma(2a+3/4)}$$

$$[t_1^a(1-t_1)^a {}_2F_1[4a, 1/2; 2a+3/4; t_1]$$

$$+ t_2^a(1-t_2)^a {}_2F_1[4a, 1/2; 2a+3/4; t_2]$$

$$+ t_3^a(1-t_3)^a \ {}_2F_1[\ 4a,\ 1/2;\ 2a+3/4;\ t_3]$$
$$+ t_4^a(1-t_4)^a \ {}_2F_1[\ 4a,\ 1/2;\ 2a+3/4;\ t_4]$$

where t_1, t_2, t_3 and t_4 are the four roots of the equation

$$(2t-1)^4 + 16t(1-t)x = 0 \tag{175}$$

as well as the Gauss-Euleur transformations.

Lemma 51. For any $a = -n/6$, $n \in N$, the following hypergeometric functions reduce to the Orthogonal Polynomials of lemma 44.

$$\begin{array}{ll} {}_2F_1[\ a,\ a+1/2;\ 2a+5/6;\ z] & {}_2F_1[\ a+5/6,\ a+1/2;\ 2a+5/6;\ z] \\ {}_2F_1[\ a,\ a+1/3;\ 2a+5/6;\ z] & {}_2F_1[\ a+5/6,\ a+1/3;\ 2a+5/6;\ z] \end{array} \tag{176}$$

Proof This is a consequense of the following transformations of sixth degree:

$$\begin{aligned} &{}_2F_1[\ 6a,\ 2/3-2a;\ 2a+5/6;\ z] \\ &= (1-16z+16z^2)^{-3a} {}_2F_1[\ a,\ a+1/3;\ 2a+5/6;\ \frac{108z(1-z)}{(1-16z+16z^2)^3}] \end{aligned} \tag{177}$$

as well as the Gauss-Euler transformations.

It is worth mentioning that quadratic transformations, like the following one:

$${}_2F_1[\ 2a,\ 2b;\ a+b+1/2;\ z] = {}_2F_1[\ a,\ b;\ a+b+1/2;\ 4z(1-z)] \tag{178}$$

can be utilized for similar purposes, and in our case, successfully. However, it turns out that linear transformations are sufficient to achieve the same goals.

The next natural step is to generalize the above lemmas. However, we will not investigate every case as we did in some previous instances before since such a task is trivial by now. Instead we will gather our accumulated results, put them in their generalized form in the following lemma and leave the reader to investigate the details.

Lemma 52. The following hypergeometric functions reduce to some Orthogonal Polynomials mentioned in Lemma 44

$$
\begin{array}{ll}
{}_2F_1[\, a+k,\ a+1/3+l;\ 1/2+m;\ z] & {}_2F_1[\, a+k,\ 1/6-a+l;\ 1/2+m;\ z] \\
{}_2F_1[\, 1/2-a+k,\ 1/6-a+l;\ 1/2+m;\ z] & {}_2F_1[\, 1/2-a+k,\ 1/3+a+l;\ 1/2+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/2+l;\ 2a+5/6+m;\ z] & {}_2F_1[\, a+k,\ a+1/3+l;\ 2a+5/6+m;\ z] \\
{}_2F_1[a-5/6+k,\ a+1/3+l;\ 2a+5/6+m;\ z] & {}_2F_1[\, a-5/6+k,\ a+1/2+l;\ 2a+5/6+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/2+l;\ 2/3+m;\ z] & {}_2F_1[\, a+k,\ 1/6-a+l;\ 2/3+m;\ z] \\
{}_2F_1[\, 2/3-a+k,\ a+1/2+l;\ 2/3+m;\ z] & {}_2F_1[\, 2/3-a+k,\ 1/6-a+l;\ 2/3+m;\ z] \\
{}_2F_1[\, a+k,\ 1/6-a+l;\ 2/3+m;\ z] & {}_2F_1[\, 2/3-a+k,\ 1/6-a+l;\ 2/3+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/2+l;\ 2/3+m;\ z] & {}_2F_1[\, 2/3-a+k,\ a+1/2+l;\ 2/3+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/4+l;\ 2a+3/4+m;\ z] & {}_2F_1[\, a+3/4+k,\ a+1/4+l;\ 2a+3/4+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/2+l;\ 2a+3/4+m;\ z] & {}_2F_1[\, a+3/4+k,\ a+1/2+l;\ 2a+3/4+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/4+l;\ 1/2+m;\ z] & {}_2F_1[\, a+k,\ 1/4-a+l;\ 1/2+m;\ z] \\
{}_2F_1[\, 1/2-a+k,\ 1/4-a+l;\ 1/2+m;\ z] & {}_2F_1[\, 1/2-a+k,\ a+1/4+l;\ 1/2+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/2+l;\ 3/4+m;\ z] & {}_2F_1[\, a+k,\ 1/4-a+l;\ 3/4+m;\ z] \\
{}_2F_1[\, 3/4-a+k,\ 1/4-a+l;\ 3/4+m;\ z] & {}_2F_1[\, 3/4-a+k,\ a+1/2+l;\ 3/4+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/2+l;\ 2a+5/6+m;\ z] & {}_2F_1[\, a+5/6+k,\ a+1/2+l;\ 2a+5/6+m;\ z] \\
{}_2F_1[\, a+k,\ a+1/3+l;\ 2a+5/6+m;\ z] & {}_2F_1[\, a+5/6+k,\ a+1/3+l;\ 2a+5/6+m;\ z]
\end{array}
\qquad (179)
$$

where k, l and m are appropriate integer numbers.

Proof This is a consequence of lemmas (44) through (51), the differentiation formulas (26) through (33) of Chapter 2 and the following differential relations

$$2^m \frac{d^m}{dx^m} P_{n,(\alpha,\beta)}(x) \tag{180}$$

$$= (n+\alpha+\beta+1)_m \, P_{n-m,(\alpha+m,\beta+m)}(x)$$

$$\frac{d^m}{dx^m} C_{n,\lambda}(x) = 2^m (\lambda)_m \, C_{n-m,\lambda+m}(x) \tag{181}$$

3.2.3.5. INCOMPLETE BETA FUNCTION REDUCTIONS

Reductions of a hypergeometric function to an Incomplete Beta function is the last to be studied in the hypergeometric function reductions.

Lemma 53. For the hypergeometric function

$${}_2F_1[\, a,\ b;\ c;\ z] \tag{182}$$

such that $a-c = -1+m, \ m \in Z^+$, the following relation holds

$${}_2F_1[\, a,\ b;\ c;\ z] \tag{183}$$

$$= \frac{1}{(1-z)^m} \left[\binom{m}{0} \frac{(b-c-(m-1))(b-c-(m-2)) \ldots (b-c)}{c(c+1)(c+2) \ldots (c+(m-1))} z^m {}_2F_1[\, a,\ b;\ c+m;\ z] \right.$$

$$+ \binom{m}{1} \frac{(b-c-(m-2))(b-c-(m-3)) \dots (b-c)}{c(c+1)(c+2) \dots (c+(m-2))} z^{m-1} {}_2F_1[\, a-1,\ b;\ c+m-1;\ z]$$

$$\dots \quad \dots \quad \dots$$

$$+ \binom{m}{m-1} \frac{b-c}{c} z \, {}_2F_1[\, a-(m-1),\ b;\ c+1;\ z]$$

$$+ \binom{m}{m} {}_2F_1[\, a-m,\ b;\ c;\ z]]$$

<u>Lemma 54.</u> For the hypergeometric function (182) such that

$$a-c = -1+m, \quad m \in Z^{-} \qquad (184)$$

the following relation holds

$${}_2F_1[\, a,\ b;\ c;\ z] \qquad (185)$$

$$= \frac{1}{(c-a-n)(c-a-(n-1)) \dots (c-a-1)}$$

$$[\binom{n}{0} (c-n)(c-(n-1)) \dots (c-1) \, {}_2F_1[\, a,\ b;\ c;\ z]$$

$$- \binom{n}{1} a(c-(n-1))(c-(n-2)) \dots (c-1) \, {}_2F_1[\, a+1,\ b;\ c-(n-1);\ z]$$

$$+ \binom{n}{2} a(a+1)(c-(n-2))(c-(n-3)) \dots (c-1) \, {}_2F_1[\, a+2,\ b;\ c-(n-2);\ z]$$

$$\dots \quad \dots \quad \dots$$

$$+ (-1)^{n+1} \binom{n}{n} a(a+1)(a+2) \dots (a+(n-1)) \, {}_2F_1[\, a+n,\ b;\ c;\ z]]$$

The following theorem summarizes the Incomplete Beta case reduction.

Theorem 5. The hypergeometric function (182) reduces to Incomplete Beta functions if either of the following conditions holds

$$a - c = 1 + m, \quad m \in Z \tag{186}$$

$$b - c = 1 + m, \quad m \in Z \tag{187}$$

3.2.4. OTHER REDUCTIONS

For higher values of p and q of the Generalized Hypergeometric Function ${}_pF_q(z)$, besides the general reduction methods that apply to them we also utilize the relations that appear in table 5 of Chapter 3.1. It turns out that these formulas are often utilized to solvs problems appearing in entries of the tables of the Bateman Manuscript.

Chapter 4

CONCLUSIONS AND FURTHER RESEARCH

Our thesis constitutes the first systematic effort towards the automation of the definite integrals for Special Functions, particularly the automation of the Bateman's Manuscript Project. Research and implementation are still under way and we feel that a lot more can be accomplished.

Let us first present some statistics of our "early" implementation of the Laplace transforms and see what this package can accomplish in comparison with the Bateman's Manuscript. Notice, that this package incorporates a proper subset of the methods presented here. The total number of formulas in Bateman is approximately 5,500. The total number of formulas that involve Special Functions in their entries in the "Laplace section" of the Bateman Manuscript is approximately 450. The total number of formulas that involve Special Functions in their entries throughout the two volumes of the Bateman Manuscript is approximately 600. Currently, we incorporate eight formulas in the table look-up. We estimate that in order to exhaust the Laplace and K transforms we need thirty to thirty five formulas. However, few are required to cover the largest parts of these sections. With eight formulas in the table we can exhaust and solve all entries in the Bateman Manuscript for Special Functions of linear or square roots of linear argument. We estimate to cover at least 75% of the

Special Functions of other arguments in the Bateman Manuscript Project (i.e. t^{-1}, t^2, e^{-t}, $(t^2+a^2)^{1/2}$, sinh(t) etc.) as soon as a Hankel implementation has been made. The rest of them will require implementations of other integral transforms. The Special Functions of other arguments occupy approximately 35% of the total number of Laplace transforms entries.

More, specifically, our current Laplace transforms implementation is generally capable of integrating expressions described in the two categories below:

1. Special Functions of linear or quadratic argument multiplied with:

   a. Arbitrary powers of the argument

   b. Trigonometric and exponential functions of linear argument.

2. Products of two Special Functions of linear or quadratic argument, multiplied with the same kind of functions we mentioned in the first category. The Special Functions of this latter category can be functions of only one of the following groups:

   a. Any kind of Bessel, Modified Bessel, or Hankel functions.

   b. Orthogonal Polynomials.

   c. Confluent Hypergeometric Functions.

The package is relatively fast, as the actual examples in the appendix 1 show. The only main external package it utilizes is the pattern matching routines of Schatchen [23].

Let us next present points of research that might be followed to increase the current capabilities of our design.

1. Computational methods to facilitate the expression of products of Generalized Hypergeometric Functions in terms of one Generalized Hypergeometric Function and vice versa. These computational methods can help both stages one and three.

2. Computational methods for the reduction of Gauss hypergeometric functions to inverse automorphic functions[3].

3. Generalizations of the ${}_pF_q(z)$ to incorporate the G-function and the MacRobert's E-function. It will basically require an additional computation in the existing reduction procedure: The reduction of the G and E functions to ${}_pF_q(z)$ wherever possible.

4. Computational methods for finding definite integrals of functions other than Special Functions (eg. algebraic etc.) resulting in expressions involving Special Functions.

5. Computational methods for the summation of the Generalized Hypergeometric Functions. This is sometimes a necessary step for those cases in which the "transform parameter" has a particular numerical value. Sometimes the existing reduction methods are sufficient particularly the general reduction methods. (see also the introduction of Chapter 3 for more comments).

We next give some ideas where our scheme - particularly stage 3 - can be utilized for other than definite integration purposes.

1. Differentiation of Special Functions. Here, we should incorporate stage 1, a couple of differentiation formulas at the Generalized Hypergeometric level for stage 2, stage 3 and the well known differentiation algorithm from Calculus. It should be noticed that differentiation does not increase the values of p and q of the ${}_pF_q(z)$, unlike the case of integration, and as a consequence stage 3 is greatly simplified. Likewise, stage 1 will be a proper subset of our stage 1 of our definite integration scheme since transform properties etc. should be ignored here.

2. Simplification of Special Functions. The deletion of stage 2 from our scheme results in a package for reducing an expression involving Special Functions to other Special Functions and/or elementary functions.

3. Differential equations. Stage 3 can be helpful to Generalized Hypergeometric series solutions of differential equations [24]. Actually, we feel that a similar strategy to that adopted for the definite integration problem can possibly be applied to the problem of solving Bessel, Legendre, Confluent etc. differential equations. All these differential equations can be viewed as particular cases of the differential equation (1) Chapter 2. The solution of this last equation which involves Generalized Hypergeometric Series then can be processed by the reduction methods of our stage 3.

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APPENDIX I.

This is a sample of some actual examples of the Laplace Transform system in MACSYMA. "Definte" is the top function that calls the integral transforms, it takes two arguments: the expression to be integrated and the variable, and assumes limits of integration from zero to infinity.

```
(C10) ASSUME(P > 0);
(D10)                              [P > 0]

(C11) SHOWTIME:TRUE$
time= 1 msec.

(C12) /* LAPLACE TRANSFORMS */

/* SOME ELEMENTARY FUNCTIONS. */

T^(1/2)*%E^(-A*T/4)*%E^(-P*T);
time= 26 msec.
                                                A T
                                       - P T - ---
                                                4
(D12)                       SQRT(T) %E

(C13) DEFINTE(%,T);

RPART FASL DSK MACSYM being loaded
loading done
          A
Is  - P - -  positive, negative, or zero?
          4

NEGATIVE;

GAMMA FASL DSK MAXOUT being loaded
loading done
time= 882 msec.
                                  SQRT(%PI)
(D13)                            ------------
                                       A 3/2
                                 2 (P + -)
                                       4

(C14) T^(3/4)*%E^(-T^2/2/B)*%E^(-P*T);
time= 25 msec.
                                         2
                                        T
                                     - --- - P T
                             3/4       2 B
(D14)                       T    %E

(C15) DEFINTE(%,T);
time= 1208 msec.
               3   7/8
(D15) 3 GAMMA(-) B
               4
```

```
                                                    2
                                                 B P
            8 SQRT(%PI) M                    (- ----)
                          5/8, - 1/4           2
 (-----------------------------------------------------------
     5/8   %I SQRT(2)    SQRT(2)           3   1/4
  3 2     (---------- + -------) GAMMA(-) B     SQRT(P)
              2             2              8

                                                  2                2
            3/8                                B P              B P
          2     SQRT(%PI) M                (- ----)             ----
                          5/8, 1/4            2                   4
 - ------------------------------------------------) %E       /4
     %I SQRT(2)     SQRT(2) 3        7   1/4
    (---------- + -------)   GAMMA(-) B     SQRT(P)
         2             2             8

(C16) T^(-1/2)*%E^(-2*A^(1/2)*T^(1/2))*%E^(-P*T);
time= 25 msec.
                          - P T - 2 SQRT(A) SQRT(T)
                        %E
(D16)                   ---------------------------
                                  SQRT(T)

(C17) DEFINTE(%,T);
time= 1018 msec.
                                         A
                A                       ---
               ---                      2 P
               2 P   SQRT(%PI) %E
(D17) SQRT(2) %E    (---------------
                        SQRT(2)

                                                                  A
                                                                 ---
                                                    SQRT(P)      2 P
                         SQRT(2) SQRT(%PI) ERF(-------) %E
                                                    SQRT(A)
                       - ------------------------------------)/SQRT(P)
                                           2

(C18) T^(1/2)*%E^(-P*T-A/T);
time= 17 msec.
                                     - P T - A/T
(D18)                        SQRT(T) %E

(C19) DEFINTE(%,T);
Is  A  positive, negative, or zero?

POSITIVE;
time= 201 msec.
                                                                                3/4
      %PI  (I        (2 SQRT(A) SQRT(P)) - I     (2 SQRT(A) SQRT(P))) A
             - 3/2                            3/2
(D19) -----------------------------------------------------------------------
                                         3 %PI   3/4
                                     SIN(-----) P
                                           2
```

```
(C20) SIN(A*T)*COSH(B*T^2)*%E^(-P*T);

HYPER FASL DSK MAXOUT being loaded
loading done
time= 21 msec.
                          - P T                 2
(D20)                   %E      SIN(A T) COSH(B T )

(C21) DEFINTE(%,T);
time= 2984 msec.
                                                      2
                      2                    (P + %I A)
            (P + %I A)                   - -----------
          - -----------                        8 B
                8 B      SQRT(%PI) %E
(D21) - %E              (-------------------------
                                  SQRT(2)

                          2
                (P + %I A)
              - -----------
                    8 B           2 %I SQRT(B)
  SQRT(%PI) %E               ERF(- ------------)
                                    P + %I A
 - --------------------------------------------)/(4 SQRT(2) SQRT(B))
                     SQRT(2)

                                                2
                                      (P + %I A)
                     2                -----------
           (P + %I A)                     8 B         2 SQRT(B)
           ----------- SQRT(%PI) %E            ERF(- ---------)
               8 B                                    P + %I A
 + %I %E             (--------------------------------------
                                   SQRT(2)

                       2
              (P + %I A)
              -----------
                  8 B
  SQRT(%PI) %E
 + -----------------------)/(4 SQRT(2) SQRT(B))
          SQRT(2)

                                                  2
                     2                  (P - %I A)
           (P - %I A)                 - -----------
         - -----------                      8 B
               8 B      SQRT(%PI) %E
 + %E                  (-------------------------
                                 SQRT(2)

                          2
                (P - %I A)
              - -----------
                    8 B           2 %I SQRT(B)
  SQRT(%PI) %E               ERF(- ------------)
                                    P - %I A
 - --------------------------------------------)/(4 SQRT(2) SQRT(B))
                     SQRT(2)
```

```
                                                            2
                                                 (P - %I A)
                          2                      -----------
             (P - %I A)                              8 B           2 SQRT(B)
             -----------  SQRT(%PI) %E                       ERF(- ---------)
                 8 B                                               P - %I A
   - %I %E               (-----------------------------------------
                                                SQRT(2)

                                       2
                   (P - %I A)
                   -----------
                       8 B
      SQRT(%PI) %E
   + -----------------------)/(4 SQRT(2) SQRT(B))
             SQRT(2)

(C22) /* SOME "CONFLUENTS". NOTICE THAT "M[K,M](Z)" IS A WHITTAKER FU
NCTION. */

%E^(A*T)*T^2*ERF(T^(1/2))*%E^(-P*T);
time= 18 msec.
                                          2   A T - P T
(D22)                  ERF(SQRT(T)) T  %E

(C23) DEFINTE(%,T);
Is  A - P  positive, negative, or zero?
NEGATIVE;
time= 417 msec.
                 1                      2
(D23) 15 (--------------- - ------------------------
                 1                          1       3/2
          SQRT(----- + 1)   3 (P - A) (----- + 1)
               P - A                    P - A

                                                 1                      7/2
                            + ------------------------)/(4 (P - A)    )
                                            2    1       5/2
                               5 (P - A)  (----- + 1)
                                           P - A

(C24) T^(1/2)*GAMMAINCOMPLETE(1/2,A*T)*%E^(-P*T);
time= 12 msec.
                                  1                    - P T
(D24)             GAMMAINCOMPLETE(-, A T) SQRT(T) %E
                                  2

(C25) DEFINTE(%,T);
time= 1702 msec.
                     %PI                                2
(D25)    ---------------------------- - ------------------------
                    3/2        A   3/2              3/2        A   3/2
         2 (P + A)     (1 - -----)     (P + A)     (1 - -----)
                             P + A                         P + A

(C26) T^(3/2)*GAMMAGREEK(3/4,A*T)*%E^(-P*T);
time= 12 msec.
                              3        3/2    - P T
(D26)               GAMMAGREEK(-, A T) T     %E
                              4
```

```
(C27) DEFINTE(%,T);
time= 276 msec.
                                           1
                                   15 GAMMA(-)
                                           4
(D27)                   -------------------------------
                            %I SQRT(2)   SQRT(2) 3  5/2
                        16 (---------- + -------)  P
                                2            2

(C28) T*M[1/2,3/4](A*T)*%E^(-P*T);
time= 12 msec.
                                               - P T
(D28)                     M         (A T) T %E
                           1/2, 3/4

(C29) DEFINTE(%,T);
time= 418 msec.
             1                        2 A    5/4    A        3/4
      15 GAMMA(-) P              (1 - -----) A    (----- - 1)
             4    - 7/4, - 3/2          A              A
                                    P + -          P + -
                                        2              2
(D29) --------------------------------------------------------
                          A 13/4        A   3/2    A        3/4
       16 SQRT(%PI) (P + -)      (1 - -----)    (----- + 1)
                          2               A            A
                                      P + -        P + -
                                          2            2

(C30) T^(3/2)*M[1/2,1](T)*%E^(-P*T);
time= 12 msec.
                                         3/2   - P T
(D30)                      M       (T) T    %E
                            1/2, 1

(C31) DEFINTE(%,T);
time= 1025 msec.
                         1                1
                  6 (--------- + ---------------------)
                           1             1         1  2
                     1 - -----   3 (P + -) (1 - -----)
                             1           2          1
                         P + -                  P + -
                             2                      2
(D31)             ------------------------------------
                                      1 4
                                  (P + -)
                                      2

(C32) /* SOME BESSEL FUNCTS (BF'S). */

/* J[V](Z), 1ST KIND OF BF'S. */

/* Y[V](Z), 2ND KIND OF BF'S.*/

/* H[V,1](Z), 1ST KIND OF THE 3RD KIND OF BF'S (1ST HANKEL). */

/* H[V,2](Z), 2ND KIND OF THE 3RD KIND OF BF'S (2ND HANKEL).*/
```

```
T^(-1/2)*J[0](2*A^(1/2)*T^(1/2))*%E^(-P*T);
time= 17 msec.
                                                         - P T
                         J (2 SQRT(A) SQRT(T)) %E
                          0
(D32)                    ----------------------------
                                   SQRT(T)

(C33) DEFINTE(%,T);
Is  A  zero or nonzero?

NONZERO;
time= 277 msec.
                                                      A
                                                    - ---
                                         A            2 P
                           SQRT(%PI) I  (---) %E
                                      0 2 P
(D33)                      ------------------------
                                   SQRT(P)

(C34)  T^(1/2)*J[1](2*A^(1/2)*T^(1/2))*%E^(-P*T);
time= 15 msec.
                                                             - P T
(D34)               J (2 SQRT(A) SQRT(T)) SQRT(T) %E
                     1

(C35) DEFINTE(%,T);
time= 221 msec.
                                                 - A/P
                                   SQRT(A) %E
(D35)                              --------------
                                           2
                                          P

(C36)  T^2*J[1](A*T)*%E^(-P*T);
time= 11 msec.
                                          2   - P T
(D36)                          J (A T) T  %E
                                1

(C37) DEFINTE(%,T);
time= 941 msec.
                                       3 A
(D37)                              -------------
                                     2
                                    A       5/2  4
                                   (-- + 1)     P
                                     2
                                    P

(C38)  T^(3/2)*Y[1](A*T)*%E^(-T);
time= 9 msec.
                                          3/2    - T
(D38)                          Y (A T) T     %E
                                1
```

```
(C39) DEFINTE(%,T);
time= 325 msec.
                                          %I     1          3/4
            15 %I SQRT(2) P            (- --) (------ - 1)
                                 - 2, 1/2   A      2
                                                  A  + 1
(D39)       -----------------------------------------------
                                   2      2     2      2     1/4
              8 SQRT(%PI) (A  + 1)   ((A  + 1)   - 1)

(C40) T^3*Y[3/4](T^(1/2))*%E^(-P*T);
time= 13 msec.
                                           3   - P T
(D40)                    Y   (SQRT(T)) T  %E
                          3/4

(C41) DEFINTE(%,T);
time= 1785 msec.
                                                 1
                                               - ---
                  3                1             8 P
        5643 GAMMA(-) M          (---) %E
                  8   7/2, 3/8 4 P
(D41) - -----------------------------------
                            3   7/2
                  512 GAMMA(-) P
                            4

                                                                  1
                                                                - ---
                                     5                 1          8 P
                  1365 SQRT(2) GAMMA(-) M           (---) %E
                                     8   7/2, - 3/8 4 P
                - -------------------------------------------
                                           1   7/2
                              256 GAMMA(-) P
                                           4

(C42) T^(4/3)*Y[3/4](T^(1/2))*%E^(-P*T);
time= 13 msec.
                                          4/3   - P T
(D42)                   Y   (SQRT(T)) T     %E
                         3/4

(C43) DEFINTE(%,T);
time= 1795 msec.
                                                 1
                                               - ---
                  17               1             8 P
        697 GAMMA(--) M          (---) %E
                  24   11/6, 3/8 4 P
(D43) - -----------------------------------
                            3   11/6
                  216 GAMMA(-) P
                            4
```

```
                                                                  1
                                                                - ---
                                       23                 1       8 P
                   23 SQRT(2) GAMMA(--) M               (---) %E
                                       24   11/6, - 3/8 4 P
                 - ----------------------------------------------
                                                1   11/6
                                     12 GAMMA(-) P
                                                4

(C44) T^(3/2)*Y[1/2](A*T)*%E^(-P*T);
time= 13 msec.
                                    3/2   - P T
(D44)                  Y    (A T) T    %E
                        1/2

(C45) DEFINTE(%,T);
time= 1274 msec.
                                                2
                             1             2 A
                 SQRT(2) (------ - ------------)
                              2            2
                             A            A       2  2
                             -- + 1    (-- + 1)  P
                              2            2
                             P            P
(D45)          - ------------------------------
                                               2
                        SQRT(%PI) SQRT(A) P

(C46) T^(3/2)*H[1/2,1](T)*%E^(-P*T);
time= 12 msec.
                                     3/2    - P T
(D46)                  H        (T) T    %E
                        1/2, 1

(C47) DEFINTE(%,T);
time= 731 msec.
                         4
(D47) ------------------------------
                           1      2  3
      SQRT(2) SQRT(%PI) (-- + 1)  P
                           2
                          P

                                      1           2
                         %I SQRT(2) (------ - ------------)
                                      1          1       2  2
                                      -- + 1   (-- + 1)  P
                                       2          2
                                      P          P
                       - ----------------------------------
                                                  2
                                      SQRT(%PI) P
```

```
(C48) T^(1/2)*H[3/4,2](T)*%E^(-P*T);
time= 12 msec.
                                           - P T
(D48)                 H       (T) SQRT(T) %E
                       3/4, 2

(C49) DEFINTE(%,T);
time= 2997 msec.
                  1                       1         1       3/8  9/4
      5 %I GAMMA(-) P              (------------) (-- - 1)     P
                  4    - 3/2, - 3/4      1         4
                                   SQRT(-- + 1)    P
                                          2
                                         P
(D49) -----------------------------------------------------------
                              %I SQRT(2)    SQRT(2)        2 3
            18 SQRT(2)  (---------- + -------) GAMMA  (-)
                                 2            2             4

           1                         1          1       3/8  9/4
   5 GAMMA(-) P               (------------) (-- - 1)     P
           4    - 3/2, - 3/4        1         4
                              SQRT(-- + 1)    P
                                     2
                                    P
 + -----------------------------------------------------------
                          %I SQRT(2)    SQRT(2)        2 3
        18 SQRT(2)  (---------- + -------) GAMMA  (-)
                             2            2             4

            3                        1          3/4
   4 %I GAMMA(-) P              (------------) P
            4    - 3/2, 3/4           1
                                SQRT(-- + 1)
                                       2
                                      P
 + -------------------------------------------
    %I SQRT(2)    SQRT(2) 3       2 1   1       3/8
   (---------- + -------)   GAMMA  (-)  (-- - 1)
         2            2               4    4
                                          P

(C50) T*H[2/3,1](T^(1/2))*%E^(-P*T);
time= 12 msec.
                                          - P T
(D50)                 H       (SQRT(T)) T %E
                       2/3, 1

(C51) DEFINTE(%,T);
time= 2308 msec.
                                               1
                                            - ---
                 1              1             8 P
        4 %I GAMMA(-) M        (---) %E
                 3    3/2, 1/3 4 P
(D51) - ------------------------------------
                                      2   3/2
            3 SQRT(3) GAMMA(-) P
                                      3
```

```
                                            1
                                          - ---
            1               1              8 P
  4 GAMMA(-) M          (---) %E
            3   3/2, 1/3 4 P
+ -----------------------------------
                        2   3/2
          3 GAMMA(-) P
                        3

                                                  1
                                                - ---
                2                 1              8 P
  8 %I GAMMA(-) M              (---) %E
                3   3/2, - 1/3 4 P
- ------------------------------------------
                             1   3/2
          3 SQRT(3) GAMMA(-) P
                             3

(C52) /* I[V](Z), K[V](Z), MODIFIED BF'S. */

T^(-1/2)*I[1](2*A^(1/2)*T^(1/2))*%E^(-P*T);
time= 17 msec.
                                                    - P T
                         I (2 SQRT(A) SQRT(T)) %E
                          1
(D52)                    -----------------------------
                                   SQRT(T)

(C53) DEFINTE(%,T);
Is  A  zero or nonzero?

NONZERO;
time= 384 msec.
                                                    A
                                                   ---
                                          A        2 P
                         SQRT(%PI) I    (---) %E
                                    1/2 2 P
(D53)                    -------------------------
                                  SQRT(P)

(C54) T^(1/2)*I[1](T)*%E^(-P*T);
time= 11 msec.
                                            - P T
(D54)                       I (T) SQRT(T) %E
                             1

(C55) DEFINTE(%,T);
time= 297 msec.
                                      1                1        5/2
         3 SQRT(%PI) P          (------------) SQRT(-- - 1) P
                       - 3/2, - 1        1            4
                                SQRT(1 - --)          P
                                          2
                                         P
(D55)    -------------------------------------------------------
                                   16
```

```
(C56) T^2*K[3/4](T^(1/2))*%E^(-P*T);
time= 13 msec.
                                        2   - P T
(D56)                  K    (SQRT(T)) T  %E
                        3/4

(C57) DEFINTE(%,T);
time= 2840 msec.
                                                           1    %I %PI
                                                          --- - ------
                             5                   1       8 P      8
      65 %PI SQRT(2) GAMMA(-) M             (- ---) %E
                             8   5/2, - 3/8   4 P
(D57) ---------------------------------------------------------------
                                      1  5/2
                            64 GAMMA(-) P
                                      4

                                                               1    7 %I %PI
                                                              --- - --------
                                   3                   1     8 P       8
           209 %PI SQRT(2) GAMMA(-) M             (- ---) %E
                                   8   5/2, 3/8    4 P
         - -------------------------------------------------------------
                                        3  5/2
                                 128 GAMMA(-) P
                                        4

(C58) T^(5/2)*K[1/2](T)*%E^(-P*T);
time= 12 msec.
                                       5/2   - P T
(D58)                    K    (T) T     %E
                          1/2

(C59) DEFINTE(%,T);
time= 1889 msec.
(D59)

                                                     4              1
3 (%I - 1) (%I + 1) SQRT(2) SQRT(%PI) (-------------- + ---------)
                                              1  3  2        1  2
                                        3 (1 - --)  P    (1 - --)
                                                2              2
                                               P              P
-----------------------------------------------------------------------
                                     4
                                  2 P

                                                     4              1
    (%I - 1) (%I + 1) SQRT(2) SQRT(%PI) (------------ + ---------)
                                              1  3  2        1  2
                                         (1 - --)  P    (1 - --)
                                               2              2
                                              P              P
 - --------------------------------------------------------------
                                      3
                                   2 P
```

```
(C60) T^3*J[0](T^(1/2))^2*%E^(-P*T);
time= 13 msec.
                              2            3   - P T
(D60)                        J (SQRT(T)) T  %E
                              0

(C61) DEFINTE(%,T);
time= 906 msec.
                               1                       1
                             - ---                   - ---
                      1        2 P             1       2 P
          9 M        (-) %E          5 M      (-) %E
             1, 1 P                     3/2, 3/2 P
(D61) 6 (------------------ - ---------------------
             16 SQRT(P)                96 P

                                                        1
                                                      - ---
                                          1           2 P                   1
                              3 M        (-) %E                           - ---
                                 1/2, 1/2 P                      1          2 P    4
                            - ---------------------- + I  (---) %E            )/P
                                          2                0 2 P

(C62) J[1](T)^2*%E^(-P*T+R);
time= 12 msec.
                                  2       R - P T
(D62)                            J (T) %E
                                  1

(C63) DEFINTE(%,T);
time= 269 msec.
                                          4         2
                                         (-- + 2) P
                                           2
                                          P                R
                 %PI SQRT(2) Q        (- -----------) %E
                              1/2, 0          4
(D63)            ------------------------------------
                                      4  3/2  3
                          16 (1 - --)      P
                                       2
                                      P

(C64) T^(1/2)*J[1/2](T^(1/2))^2*%E^(-P*T);
time= 15 msec.
                           2                          - P T
(D64)                     J    (SQRT(T)) SQRT(T) %E
                           1/2

(C65) DEFINTE(%,T);
time= 299 msec.
                                                   - 1/P
                           %I ERF (%I SQRT(P)) %E
(D65)                      -------------------------
                                                3/2
                                 SQRT(%PI) P

(C66) T^(5/2)*Y[1/2](T^(1/2))^2*%E^(-P*T);
```

```
time= 15 msec.
                           2              5/2   - P T
(D66)                     Y    (SQRT(T)) T     %E
                           1/2

(C67) DEFINTE(%,T);
time= 1599 msec.
                                               1                        1
                                             - ---                    - ---
                              1   1/4          2 P          1           2 P
               2 M           (-) P      %E          4 M    (-) %E
                  3/4, 3/4 P                            5/4, 5/4 P
(D67) - 12 (- -------------------------- + ---------------------
                            3                                1/4
                                                          45 P

                                                        - 1/P
              %I SQRT(%PI) ERF(%I SQRT(P)) SQRT(P) %E                  4
            + ---------------------------------------------)/(%PI P )
                                  2

(C68) I[0](2*A^(1/2)*T^(1/2))^2*%E^(-P*T);
time= 15 msec.
                           2                           - P T
(D68)                     I (2 SQRT(A) SQRT(T)) %E
                           0

(C69) DEFINTE(%,T);
Is  A  zero or nonzero?

NONZERO;
time= 300 msec.
                                            2 A
                                            ---
                                     2 A     P
                                  I (---) %E
                                   0  P
(D69)                             -------------
                                        P

(C70) T^(3/4)*J[1/2](T)*J[1/4](T)*%E^(-P*T);
time= 15 msec.
                                           3/4   - P T
(D70)                     J    (T) J    (T) T     %E
                           1/4      1/2

(C71) DEFINTE(%,T);
time= 1266 msec.
                                        1           16     1/4
                3 P               (-------------) (-- - 1)     P
                   - 5/4, - 1/2          4           4
                                   SQRT(-- + 1)     P
                                          2
                                         P
(D71)       - -------------------------------------------------
                3/4  %I SQRT(2)    SQRT(2) 3                  1
             4 2    (---------- + -------)  SQRT(%PI) GAMMA(-)
                         2            2                       4

(C72) J[1/2](T^(1/2))*Y[1/2](T^(1/2))*%E^(-P*T);
```

```
time= 15 msec.
                                                      - P T
(D72)            J    (SQRT(T)) Y    (SQRT(T)) %E
                  1/2              1/2

(C73) DEFINTE(%,T);
time= 365 msec.
                                                   1
                                                 - ---
                                          1        2 P
                                %I  I    (---) %E
                                    1/2 2 P
(D73)                         - -------------------
                                          P

(C74) T*I[0](A*T/2)*I[1](A*T/2)*%E^(-P*T);
time= 17 msec.
                                A T     A T       - P T
(D74)                        I (---) I (---) T %E
                              0  2    1  2

(C75) DEFINTE(%,T);
time= 1203 msec.
                                           2           2
                                       2 A           A
                      P           (1 - ----) A SQRT(-- - 1)
                       - 1/2, - 1       2            2
                                       P            P
(D75)                 -----------------------------------
                                    2        2
                                  A        A        3
                          2 (1 - --) SQRT(-- + 1) P
                                  2        2
                                 P        P

(C76) I[1/2](T^(1/2))*K[1/2](T^(1/2))*%E^(-P*T);
time= 15 msec.
                                                      - P T
(D76)            I    (SQRT(T)) K    (SQRT(T)) %E
                  1/2              1/2

(C77) DEFINTE(%,T);
time= 2389 msec.
                                            1                              1
                                           ---                            ---
                                   1       2 P                    1       2 P
       %I %PI (%I + 1) I     (---) %E        %PI (%I + 1) I     (---) %E
                          1/2 2 P                          1/2 2 P
(D77)  ------------------------------- - ---------------------------
                      4 P                             4 P

                                            1                              1
                                           ---                            ---
                                   1       2 P                    1       2 P
        %I %PI (%I - 1) I     (---) %E       %PI (%I - 1) I     (---) %E
                           1/2 2 P                         1/2 2 P
     + ------------------------------- + ---------------------------
                      4 P                             4 P

(C78) /* RELATED TO BF'S FUNCTIONS. */
```

```
/* STRUVE FUNCTIONS. */

T^(-1/2)*LSTRUVE[-1/2](T^(1/2))*%E^(-P*T);
time= 16 msec.
                                             - P T
                   LSTRUVE      (SQRT(T)) %E
                          - 1/2
(D78)              ----------------------------
                              SQRT(T)

(C79) DEFINTE(%,T);
time= 1333 msec.
                                                                     1
                                                                    ---
                                         1       3         1        8 P
        (%I - 1) (%I + 1) SQRT(2) GAMMA(-) GAMMA(-) I    (---) %E
                                         4       4   1/4 8 P
(D79) - ----------------------------------------------------------
                              4 SQRT(%PI) SQRT(P)

(C80) T^(3/2)*HSTRUVE[1](T^(1/2))*%E^(-P*T);
time= 13 msec.
                                          3/2   - P T
(D80)                  HSTRUVE (SQRT(T)) T    %E
                              1

(C81) DEFINTE(%,T);
time= 743 msec.
                                                  1
                                                - ---
                                   1   3/4        8 P
             16 SQRT(2) M        (---) P     %E
                         1/4, 5/4 4 P
(D81) 5 (- ------------------------------------
                             15

                                           1
                                         - ---
                             1    1/4      8 P
   32 SQRT(2) M           (---) P     %E
                3/4, 7/4 4 P
 + ------------------------------------
                      525

                                              1
                                            - ---
                           3     1    3/2     4 P             7/2
 + 12 %I GAMMAGREEK(-, - ---) P     %E        )/(3 %PI P    )
                           2     4 P

(C82) T^(-1/2)*LSTRUVE[-1/2](A*T)*%E^(-P*T);
time= 16 msec.
                                           - P T
                       LSTRUVE      (A T) %E
                              - 1/2
(D82)                  -------------------------
                                SQRT(T)

(C83) DEFINTE(%,T);
time= 1279 msec.
```

```
                                                             %I P
                  %I (%I - 1) (%I + 1) SQRT(2) ERF(----)
                                                              A
(D83)          - ------------------------------------
                                  4 SQRT(A)

(C84) T*HSTRUVE[1](T)*%E^(-P*T);
time= 10 msec.
                                                - P T
(D84)                       HSTRUVE (T) T %E
                                   1

(C85) DEFINTE(%,T);
time= 232 msec.
                                       16 %I
(D85)                      - -----------------------
                                  3/2  1       3/2  3
                             3 %PI    (-- + 1)    P
                                        2
                                       P

(C86) /* LOMMEL FUNCTIONS. */

T^(9/8)*S[1/2,1/4](T^(1/2))*%E^(-P*T);
time= 14 msec.
                                              9/8   - P T
(D86)                    S          (SQRT(T)) T    %E
                          1/2, 1/4

(C87) DEFINTE(%,T);
time= 1326 msec.
                                                         1    5 %I %PI
                                                      - --- - --------
                                  5     1    5/8       4 P       8
                        5 GAMMAGREEK(-, - ---) P    %E
                   7              8    4 P
(D87) 3 GAMMA(-) (-------------------------------------------------
                   8                            3/4
                                             2 2

                                                               1
                                                            - ---
                        5/8                   1    5/16     8 P
                     64 2    M              (---) P     %E
                               11/16, 13/16 4 P                        23/8
                     - ---------------------------------------)/(4 P     )
                                           195

(C88) T^(1/4)*S[1/2,-1/2](T^(1/2))*%E^(-P*T);
time= 16 msec.
                                              1/4   - P T
(D88)                   S           (SQRT(T)) T    %E
                         1/2, - 1/2
```

```
(C89) DEFINTE(%,T);
time= 229 msec.
                                                                   1
                                                                 - ---
                                                                   4 P
                %I SQRT(%PI) ERF(- 2 %I SQRT(P)) %E
(D89)         - ---------------------------------------------
                                               3/2
                                            2 P

(C90) T^(1/8)*SLOMMEL[1/2,1/4](T^(1/2))*%E^(-P*T);
time= 14 msec.
                                                          1/8    - P T
(D90)               SLOMMEL           (SQRT(T)) T     %E
                           1/2, 1/4

(C91) DEFINTE(%,T);
time= 5376 msec.
                                      1     %I %PI
                                     --- + ------
                  5         7        4 P      8
          GAMMA(-) GAMMA(-) %E
                  8         8
(D91) - --------------------------------
                           3/4  5/4
                        2 2     P

                               1     %I %PI
                              --- - ------
            5         7       4 P      8
     GAMMA(-) GAMMA(-) %E
            8         8
 - --------------------------------
                    3/4  5/4
                 2 2     P

                                %I %PI    1
                                ------ - ---
               5         7        8      4 P
     %I GAMMA(-) GAMMA(-) %E
               8         8
 - ----------------------------------
                      3/4  5/4
                   2 2     P

                                   1     %I %PI
                               - --- - ------
               5         7       4 P      8
     %I GAMMA(-) GAMMA(-) %E
               8         8
 + ------------------------------------
                       3/4  5/4
                    2 2     P
```

```
                                                                 1    5 %I %PI
                                                               - --- - --------
   1/4               5      1           7        4 P       8
  2     GAMMAGREEK(-,  - ---) GAMMA(-) %E
                     8    4 P           8
 + ------------------------------------------------------------
                                           5/4
                                        2 P

(C92) T^(1/4)*SLOMMEL[1/2,-1/2](T^(1/2))*%E^(-P*T);
time= 16 msec.
                                                        1/4    - P T
(D92)              SLOMMEL              (SQRT(T)) T     %E
                               1/2, - 1/2

(C93) DEFINTE(%,T);
time= 633 msec.
                                                              1
                                                            - ---
                                                              4 P
         %I SQRT(%PI) ERF(- 2 %I SQRT(P)) %E
(D93) - ------------------------------------------
                                          3/2
                                      2 P

                                                                                  1
                                                                                - ---
                                                                 1      1         4 P
                                                 %I GAMMAGREEK(- -, - ---) %E
                                                                 2    4 P
                                               - ------------------------------------
                                                                   3/2
                                                                4 P

time= 66809 msec.
(D94)                           BATCH DONE

(C95) CLOSEFILE(MUC,DEMO);
```

APPENDIX 2.

The quadratic transformations (see also [3] and [15]):

(1) $F(a,\ b;\ a-b+1;\ z)$
$$= (1-z)^{-a} F[\tfrac{1}{2}a,\ -b+(a+1)/2;\ 1+a-b;\ -4z(1-z)^{-2}]$$

(2) $F(2a,\ 2b;\ a+b+\tfrac{1}{2};\ z) = F[a,\ b;\ a+b+\tfrac{1}{2};\ 4z(1-z)]$

(3) $$F(2a,\ 2b;\ a+b+\tfrac{1}{2};\ \tfrac{1}{2}+\tfrac{1}{2}z) = \frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})} F(a,\ b;\ \tfrac{1}{2};\ z^2)$$
$$- z\,\frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(a)\,\Gamma(b)} F(a+\tfrac{1}{2},\ b+\tfrac{1}{2};\ 3/2;\ z^2)$$

(4) $F(a,\ b;\ 2b;\ z) = (1-\tfrac{1}{2}z)^{-a} F\{\tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a;\ b+\tfrac{1}{2};\ [z/(2-z)]^2\}$

(5) $F[a,\ b;\ 2b;\ 4z(1+z)^{-2}] = (1+z)^{2a} F(a,\ a+\tfrac{1}{2}-b;\ b+\tfrac{1}{2};\ z^2)$

(6) $F(a,\ a+\tfrac{1}{2};\ b;\ 2z-z^2) = (1-\tfrac{1}{2}z)^{-2a} F[2a,\ 2a-b+1;\ b;\ z/(2-z)]$

Goursat's table of quadratic transformations. The square roots are defined in such a way that their value becomes real and positive if z is real and $0 \le z < 1$. All formulas are valid in a neighborhood of $z = 0$.

(7) $$\frac{2\,\Gamma(\frac{1}{2})\,\Gamma(a+b+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})} F(a,\ b;\ \tfrac{1}{2};\ z)$$
$$= F[2a,\ 2b;\ a+b+\tfrac{1}{2};\ \tfrac{1}{2}(1+z^{\frac{1}{2}})]$$
$$+ F[2a,\ 2b;\ a+b+\tfrac{1}{2};\ \tfrac{1}{2}(1-z^{\frac{1}{2}})]$$

(8) $$\frac{2\,\Gamma(\frac{1}{2})\,\Gamma(a+1-b)}{\Gamma(a+\frac{1}{2})\,\Gamma(1-b)} (1+z)^a F(a,\ b;\ \tfrac{1}{2};\ -z)$$
$$= F[2a,\ 1-2b;\ a+1-b;\ \tfrac{1}{2}+\tfrac{1}{2}z^{\frac{1}{2}}(1+z)^{-\frac{1}{2}}]$$
$$+ F[2a,\ 1-2b;\ a+1-b;\ \tfrac{1}{2}-\tfrac{1}{2}z^{\frac{1}{2}}(1+z)^{-\frac{1}{2}}]$$

(9) $$\frac{2\,\Gamma(-\frac{1}{2})\,\Gamma(a+b-\frac{1}{2})}{\Gamma(a-\frac{1}{2})\,\Gamma(b-\frac{1}{2})} z^{\frac{1}{2}} F(a,\ b;\ 3/2;\ z)$$
$$= F(2a-1,\ 2b-1;\ a+b-\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}z^{\frac{1}{2}})$$
$$- F(2a-1,\ 2b-1;\ a+b-\tfrac{1}{2};\ \tfrac{1}{2}+\tfrac{1}{2}z^{\frac{1}{2}})$$

(10) $F(a,\ b;\ a+b+\tfrac{1}{2};\ z) = F[2a,\ 2b;\ a+b+\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}(1-z)^{\frac{1}{2}}]$

(11) $F(a,\ b;\ a+b+\tfrac{1}{2};\ z)$
$$= [\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{\frac{1}{2}}]^{-2a} F\left[2a,\ a-b+\tfrac{1}{2};\ a+b+\tfrac{1}{2};\ \frac{(1-z)^{\frac{1}{2}}-1}{(1-z)^{\frac{1}{2}}+1}\right]$$

(12) $F(a,\ b;\ a+b+\tfrac{1}{2};\ -z)$
$$= [(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2a} F[2a,\ a+b;\ 2a+2b;\ 2(z+z^2)^{\frac{1}{2}}-2z]$$

(13) $F(a,\ b;\ a+b-\tfrac{1}{2};\ z)$
$$= (1-z)^{-\frac{1}{2}} F[2a-1,\ 2b-1;\ a+b-\tfrac{1}{2};\ \tfrac{1}{2}-\tfrac{1}{2}(1-z)^{\frac{1}{2}}]$$

(next page cont'd)

(the quadratic transformations cont'd)

(14) $F(a, b; a+b-\frac{1}{2}; z) = (1-z)^{-\frac{1}{2}}[\frac{1}{2}+\frac{1}{2}(1-z)^{\frac{1}{2}}]^{1-2a}$
$\times F\left[2a-1, a-b+\frac{1}{2}; a+b-\frac{1}{2}; \frac{(1-z)^{\frac{1}{2}}-1}{(1-z)^{\frac{1}{2}}+1}\right]$

(15) $F(a, b; a+b-\frac{1}{2}; -z) = (1+z)^{-\frac{1}{2}}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{1-2a}$
$\times F[2a-1, a+b-1; 2a+2b-2; 2(z+z^2)^{\frac{1}{2}}-2z]$

(16) $F(a, a+\frac{1}{2}; c; z)$
$= (1-z)^{-a} F[2a, 2c-2a-1; c; \frac{1}{2}-\frac{1}{2}(1-z)^{-\frac{1}{2}}]$

(17) $F(a, a+\frac{1}{2}; c; z)$
$= (1+z^{\frac{1}{2}})^{-2a} F[2a, c-\frac{1}{2}; 2c-1; 2z^{\frac{1}{2}}(1+z^{\frac{1}{2}})^{-1}]$

(18) $F[a, b; (a+b+1)/2; z] = F[\frac{1}{2}a, \frac{1}{2}b; (a+b+1)/2; 4z(1-z)]$

(19) $F[a, b; (a+b+1)/2; z]$
$= (1-2z) F[\frac{1}{2}+\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}b; (a+b+1)/2; 4z(1-z)]$

(20) $F[a, b; (a+b+1)/2; z] = (1-2z)^{-a}$
$= F[\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a; (a+b+1)/2; 4z(z-1)(2z-1)^{-2}]$

(21) $F[a, b; (a+b+1)/2; -z] = [(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2a}$
$= F\{a, \frac{1}{2}a+\frac{1}{2}b; a+b; 4z^{\frac{1}{2}}(z+1)^{\frac{1}{2}}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2}\}$

(22) $F(a, 1-a; c; z)$
$= (1-z)^{c-1} F[\frac{1}{2}c-\frac{1}{2}a, (c+a-1)/2; c; 4z(1-z)]$

(23) $= (1-z)^{c-1}(1-2z) F[\frac{1}{2}c+\frac{1}{2}a, (c+1-a)/2; c; 4z(1-z)]$

(24) $F(a, 1-a; c; z) = (1-z)^{c-1}(1-2z)^{a-c}$
$\times F[\frac{1}{2}c-\frac{1}{2}a, (c+1-a)/2; c; 4z(z-1)(1-2z)^{-2}]$

(25) $F(a, 1-a; c; -z) = (1+z)^{c-1}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{2-2a-2c}$
$\times F\{c+a-1, c-\frac{1}{2}; 2c-1; 4z^{\frac{1}{2}}(1+z)^{\frac{1}{2}}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2}\}$

(26) $F(a, b; 2b; z)$
$= (1-z)^{-\frac{1}{2}a} F[\frac{1}{2}a, b-\frac{1}{2}a; b+\frac{1}{2}; (z^2/4)(z-1)^{-1}]$

(27) $= (1-\frac{1}{2}z)(1-z)^{-\frac{1}{2}a-\frac{1}{2}} F[b+\frac{1}{2}-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a; b+\frac{1}{2}; z^2/(4z-4)]$

(28) $= (1-\frac{1}{2}z)^{-a} F[\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a; b+\frac{1}{2}; z^2(2-z)^{-2}]$

(29) $= (1-z)^{b-a}(1-\frac{1}{2}z)^{a-2b} F[b-\frac{1}{2}a, b+\frac{1}{2}-\frac{1}{2}a; b+\frac{1}{2}; z^2/(2-z)^{-2}]$

(next page cont'd)

(30) $F(a, b; 2b; z) = (1-z)^{-\frac{1}{2}a}$
$\times F\{a, 2b-a; b+\frac{1}{2}; (-\frac{1}{4})(1-z)^{-\frac{1}{2}}[1-(1-z)^{\frac{1}{2}}]^2\}$

(31) $F(a, b; 2b; z) = [\frac{1}{2}+\frac{1}{2}(1-z)^{\frac{1}{2}}]^{-2a}$
$$\times F\left\{a, a-b+\tfrac{1}{2}; b+\tfrac{1}{2}; \left[\frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}}\right]^2\right\}$$

(32) $F(a, b; a-b+1; z) = (1-z)^{-a}$
$\times F[\frac{1}{2}a, (a+1-2b)/2; a-b+1; -4z(1-z)^{-2}]$

(33) $F(a, b; a-b+1; z) = (1+z)(1-z)^{-a-1}$
$\times F[\frac{1}{2}+\frac{1}{2}a, \frac{1}{2}a+1-b; a-b+1; -4z(1-z)^{-2}]$

(34) $F(a, b; a-b+1; z) = (1+z)^{-a}$
$\times F[\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; a-b-1;\ 4z(1+z)^{-2}]$ (*)

(35) $F(a, b; a-b+1; z) = (1-z)^{1-2b}(1+z)^{2b-a-1}$
$\times F[(a+1-2b)/2, (a-2b+2)/2; a+1-b; 4z(1+z)^{-2}]$

(36) $F(a, b; a-b+1; z) = (1+z^{\frac{1}{2}})^{-2a}$
$\times F[a, a-b+\frac{1}{2}; 2a-2b+1; 4z^{\frac{1}{2}}(1+z^{\frac{1}{2}})^{-2}]$

(*) The lhs hypergeometric function of the quadratic transformation (34) should be: $F(1/2a,\ 1/2a+1/2;\ a-b+1;\ 4z(1+z)^{-2})$. This is a typo error in both [3] and [15] references.

The rational cubic transformations. For an extensive list of cubic transformations see [15].

(40) $F(3a, 3a+\frac{1}{2}; 4a+2/3; z) = (1-9z/8)^{-2a}$
$\times F[a, a+\frac{1}{2}; a+5/6; -27z^2(1-z)(9z-8)^{-2}]$

(41) $F(3a, 3a+\frac{1}{2}; 2a+5/6; z) = (1-9z)^{-2a}$
$\times F[a, a+\frac{1}{2}; 2a+5/6; -27z(1-z)^2(1-9z)^{-2}]$

(42) $F(3a, a+1/6; 4a+2/3; z) = (1-z/4)^{-3a}$
$\times F[a, a+1/3; 2a+5/6; -27z^2(z-4)^{-3}]$

(43) $F(3a, 1/3-a; 2a+5/6; z) = (1-4z)^{-3a}$
$\times F[a, a+1/3; 2a+5/6; 27z(4z-1)^{-3}]$

(44) $F(3a, 1/3-a; 1/2; z) = (1-z)^{-a}$
$\times F[a, 1/6-a; 1/2; (z/27)(9-8z)^2(1-z)^{-1}]$

(45) $F(3a, a+1/6; 1/2; z) = (1-z)^{-2a}$
$\times F[a, 1/6-a; 1/2; -(z/27)(z-9)^2(1-z)^{-2}]$

(46) $F(3a+1/2, 5/6-a; 3/2; z) = (1-8z/9)(1-4z/3)^{-3a-3/2}$
$\times F[a+1/2, a+5/6; 3/2; z(9-8z)^2(4z-3)^{-3}]$

(47) $F(3a,+1/2, a+2/3; 3/2; z) = (1-z/9)(1+z/3)^{-3a-3/2}$
$\times F[a+1/2, a+5/6; 3/2; z(z-9)^2(z+3)^{-3}]$.

APPENDIX 3

The representation of elementary functions in terms of a hypergeometric function:

From reference [3]:

(4) $(1+z)^{a} = F(-a, b; b; -z)$

(5) $\frac{1}{2}(1+z^{1/2})^{-2a} + \frac{1}{2}(1-z^{1/2})^{-2a} = F(a, a+\frac{1}{2}; \frac{1}{2}; z)$

(6) $[\frac{1}{2} + (1-z)^{1/2}/2]^{1-2a} = F(a-\frac{1}{2}, a; 2a; z)$
$= (1-z)^{1/2} F(a, a+\frac{1}{2}; 2a; z)$

(7) $(1-z)^{-2a-1}(1+z) = F(2a, a+1; a; z)$

The truncated binomial series follow:

(8) $1+\binom{a}{1} z + \cdots + \binom{a}{m} z^{m} = \binom{a}{m} z^{m} \; F(-m, 1; a-m+1; -z^{-1})$

(9) $\sum_{n=m+1}^{\infty} \binom{a}{n} z^{n} = z^{m+1} \frac{\Gamma(a+1)}{\Gamma(a-m)\,(m+1)\,!} F(m+1-a, 1; m+2; -z)$

(10) $e^{-az} = (2\cosh z)^{-a} \tanh z \, F[1+\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a; 1+a; (\cosh z)^{-2}]$

(11) $\cos az = F[\frac{1}{2}a, -\frac{1}{2}a; \frac{1}{2}; (\sin z)^{2}]$
$= \cos z \, F[\frac{1}{2}+\frac{1}{2}a, \frac{1}{2}-\frac{1}{2}a; \frac{1}{2}; (\sin z)^{2}]$
$= (\cos z)^{a} \, F[-\frac{1}{2}a, \frac{1}{2}-\frac{1}{2}a; \frac{1}{2}; -(\tan z)^{2}]$

(12) $\sin az = a \sin z \, F[\frac{1}{2}+\frac{1}{2}a, \frac{1}{2}-\frac{1}{2}a; 3/2; (\sin z)^{2}]$
$= a \sin z \cos z \, F[1+\frac{1}{2}a, 1-\frac{1}{2}a; 3/2; (\sin z)^{2}]$

(13) $\sin^{-1} z = z \, F(\frac{1}{2}, \frac{1}{2}; 3/2; z^{2})$

(14) $\tan^{-1} z = z \, F(\frac{1}{2}, 1; 3/2; -z^{2})$

(15) $\log(z+1) = z \, F(1, 1; 2; -z)$

(16) $\log \frac{1+z}{1-z} = 2z \, F(\frac{1}{2}, 1; 3/2; z^{2})$

From reference [21]:

15.1.3 $F(1,1;2;z)=-z^{-1}\ln(1-z)$

15.1.4 $F(\frac{1}{2},1;\frac{3}{2};z^2)=\frac{1}{2}z^{-1}\ln\left(\frac{1+z}{1-z}\right)$

15.1.5 $F(\frac{1}{2},1;\frac{3}{2};-z^2)=z^{-1}\arctan z$

15.1.6

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2)=(1-z^2)^{1/2}F(1,1;\tfrac{3}{2};z^2)=z^{-1}\arcsin z$$

15.1.7

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-z^2)=(1+z^2)^{1/2}F(1,1;\tfrac{3}{2};-z^2)=z^{-1}\ln[z+(1+z^2)^{1/2}]$$

15.1.8 $F(a,b;b;z)=(1-z)^{-a}$

15.1.9 $F(a,\frac{1}{2}+a;\frac{1}{2};z^2)=\frac{1}{2}[(1+z)^{-2a}+(1-z)^{-2a}]$

15.1.10

$$F(a,\tfrac{1}{2}+a;\tfrac{3}{2};z^2)=\tfrac{1}{2}z^{-1}(1-2a)^{-1}[(1+z)^{1-2a}-(1-z)^{1-2a}]$$

15.1.11

$$F(-a,a;\tfrac{1}{2};-z^2)=\tfrac{1}{2}\{[(1+z^2)^{1/2}+z]^{2a}+[(1+z^2)^{1/2}-z]^{2a}\}$$

15.1.12

$$F(a,1-a;\tfrac{1}{2};-z^2)=\tfrac{1}{2}(1+z^2)^{-1/2}\{[(1+z^2)^{1/2}+z]^{2a-1}+[(1+z^2)^{1/2}-z]^{2a-1}\}$$

15.1.13

$$F(a,\tfrac{1}{2}+a;1+2a;z)=2^{2a}[1+(1-z)^{1/2}]^{-2a}=(1-z)^{1/2}F(1+a,\tfrac{1}{2}+a;1+2a;z)$$

15.1.14

$$F(a,\tfrac{1}{2}+a;2a;z)=2^{2a-1}(1-z)^{-1/2}[1+(1-z)^{1/2}]^{1-2a}$$

15.1.15 $F(a,1-a;\frac{3}{2};\sin^2 z)=\dfrac{\sin[(2a-1)z]}{(2a-1)\sin z}$

15.1.16 $F(a,2-a;\frac{3}{2};\sin^2 z)=\dfrac{\sin[(2a-2)z]}{(a-1)\sin(2z)}$

15.1.17 $F(-a,a;\frac{1}{2};\sin^2 z)=\cos(2az)$

15.1.18 $F(a,1-a;\frac{1}{2};\sin^2 z)=\dfrac{\cos[(2a-1)z]}{\cos z}$

15.1.19 $F(a,\frac{1}{2}+a;\frac{1}{2};-\tan^2 z)=\cos^{2a}z\cos(2az)$

From reference [22]:

9.121

1. $F(-n,\ \beta;\ \beta;\ -z)=(1+z)^n$ [β arbitrary] — EH I 101(4), GA 127 Ia

2. $F\left(-\frac{n}{2},\ -\frac{n-1}{2};\ \frac{1}{2};\ \frac{z^2}{t^2}\right)=\frac{(t+z)^n+(t-z)^n}{2t^n}.$ — GA 127 II

3. $\lim\limits_{\omega\to\infty} F\left(-n,\ \omega;\ 2\omega;\ -\frac{z}{t}\right)=\left(1+\frac{z}{2t}\right)^n.$ — GA 127 IIIa

4. $F\left(-\frac{n-1}{2},\ -\frac{n-2}{2};\ \frac{3}{2};\ \frac{z^2}{t^2}\right)=\frac{(t+z)^n-(t-z)^n}{2nzt^{n-1}}.$ — GA 127 IV

5. $F\left(1-n,\ 1;\ 2;\ -\frac{z}{t}\right)=\frac{(t+z)^n-t^n}{nzt^{n-1}}.$ — GA 127 V

6. $F(1,\ 1;\ 2;\ -z)=\frac{\ln(1+z)}{z}.$ — GA 127 VI

7. $F\left(\frac{1}{2},\ 1;\ \frac{3}{2};\ z^2\right)=\frac{\ln\frac{1+z}{1-z}}{2z}.$ — GA 127 VII

8. $\lim\limits_{k\to\infty} F\left(1,\ k;\ 1;\ \frac{z}{k}\right)=1+z\lim\limits_{k\to\infty} F\left(1,\ k;\ 2;\ \frac{z}{k}\right)=$
$=1+z+\frac{z^2}{2}\lim\limits_{k\to\infty} F\left(1,\ k;\ 3;\ \frac{z}{k}\right)=\ldots=e^z.$ — GA 127 VIII

9. $\lim\limits_{\substack{k\to\infty\\ k'\to\infty}} F\left(k,\ k';\ \frac{1}{2};\ \frac{z^2}{4kk'}\right)=\frac{e^z+e^{-z}}{2}=\operatorname{ch} z.$ — GA 127 IX

10. $\lim\limits_{\substack{k\to\infty\\ k'\to\infty}} F\left(k,\ k';\ \frac{3}{2};\ \frac{z^2}{4kk'}\right)=\frac{e^z-e^{-z}}{2z}=\frac{\operatorname{sh} z}{z}.$ — GA 127 X

11. $\lim\limits_{\substack{k\to\infty\\ k'\to\infty}} F\left(k,\ k';\ \frac{3}{2};\ -\frac{z^2}{4kk'}\right)=\frac{\sin z}{z}.$ — GA 127 XI

12. $\lim\limits_{\substack{k\to\infty\\ k'\to\infty}} F\left(k,\ k';\ \frac{1}{2};\ -\frac{z^2}{4kk'}\right)=\cos z.$ — GA 127 XII

13. $F\left(\frac{1}{2},\ \frac{1}{2};\ \frac{3}{2};\ \sin^2 z\right)=\frac{z}{\sin z}.$ — GA 127 XIII

14. $F\left(1,\ 1;\ \frac{3}{2};\ \sin^2 z\right)=\frac{z}{\sin z\cos z}.$ — GA 127 XIV

15. $F\left(\frac{1}{2},\ 1;\ \frac{3}{2};\ -\operatorname{tg}^2 z\right)=\frac{z}{\operatorname{tg} z}.$ — GA 127 XV

16. $F\left(\frac{n+1}{2},\ -\frac{n-1}{2};\ \frac{3}{2};\ \sin^2 z\right)=\frac{\sin nz}{n\sin z}.$ — GA 127 XVI

17. $F\left(\frac{n+2}{2},\ -\frac{n-2}{2};\ \frac{3}{2};\ \sin^2 z\right)=\frac{\sin nz}{n\sin z\cos z}.$ — GA 127 XVII

18. $F\left(-\frac{n-2}{2},\ -\frac{n-1}{2};\ \frac{3}{2};\ -\operatorname{tg}^2 z\right)=\frac{\sin nz}{n\sin z\cos^{n-1} z}.$ — GA 127 XVIII

19. $F\left(\frac{n+2}{2},\ \frac{n+1}{2};\ \frac{3}{2};\ -\operatorname{tg}^2 z\right)=\frac{\sin nz\cos^{n+1} z}{n\sin z}.$ — GA 127 XIX

20. $F\left(\frac{n}{2},\ -\frac{n}{2};\ \frac{1}{2};\ \sin^2 z\right)=\cos nz.$ — EH I 101(11), GA 127 XX

21. $F\left(\frac{n+1}{2},\ -\frac{n-1}{2};\ \frac{1}{2};\ \sin^2 z\right)=\frac{\cos nz}{\cos z}.$ — EH I 101(11), GA 127 XXI

22. $F\left(-\frac{n}{2},\ -\frac{n-1}{2};\ \frac{1}{2};\ -\operatorname{tg}^2 z\right)=\frac{\cos nz}{\cos^n z}.$ — EH I 101(11), GA 127 XXII

23. $F\left(\frac{n+1}{2},\ \frac{n}{2};\ \frac{1}{2};\ -\operatorname{tg}^2 z\right)=\cos nz\cos^n z.$ — GA 127 XXIII

24. $F\left(\frac{1}{2},\ 1;\ 2;\ 4z(1-z)\right)=\frac{1}{1-z}$ $\left[|z|\leq\frac{1}{2};\ |z(1-z)|\leq\frac{1}{4}\right].$

25. $F\left(\frac{1}{2},\ 1;\ 1;\ \sin^2 z\right)=\sec z.$

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26. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z}$ (cf. 9.121 13.).

27. $F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arctg} z}{z}$ (cf. 9.121 15.).

28. $F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{Arsh} z}{z}$ (cf. 9.121 26.).

29. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz}$ (cf. 9.121 16.).

30. $F\left(1+\frac{n}{2}, 1-\frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}}$ (cf. 9.121 17.).

31. $F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z)$ (cf. 9.121 20.).

32. $F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}$ (cf. 9.121 21.).